Extending Figlewski's option pricing formula[0]

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Extending Figlewski's option pricing formula

One of the uses of an option pricing model is to infer the price of an option from the market price of a "nearby" option. For example, given the Black-Scholes option pricing formula and the market price of an option it is possible to calculate the Black-Scholes implied volatility. This volatility can be substituted back into the Black-Scholes pricing formula to give the price of any other derivative.

As Figlewski (2002) has pointed out, if the option pricing model is to be used in this way then there is nothing special about the Black-Scholes equation and any function with the right shape, could in principle be used instead. Figlewski suggests a simple alternative function.

Unfortunately his proposed function violates static arbitrage. We suggest a simple modification which corrects for this deficiency. We also show how to incorporate maturity into the pricing model. Once maturity is included in the model it is possible to infer the dynamics of the underlying which are consistent with the pricing equations.

We also undertake a numerical investigation of the fit of both the Figlewski model and our modified version. In doing so, we often reach the same conclusions as Figlewski, but interestingly, we also sometimes find the opposite results.

The Black and Scholes (1973) model for option pricing is the industry standard and won its inventors a Nobel prize. Despite its widespread use, the theoretical underpinnings of the model are often violated in practice. Volatility is not constant, and is widely documented to exhibit smiles and skews, see Rubinstein (1985).

One of the uses of an option pricing model is to infer an option price from market prices of "nearby" options, perhaps involving a similar strike or time to maturity. In the Black and Scholes (1973) model this is accomplished via implied volatility. For example yesterday's implied volatility might be used to compute an option price today or an option price might be calculated from interpolation between implied volatilities of two options with strikes spanning the strike of interest.

The recent paper of Figlewski (2002) recognizes that this usage of the Black-Scholes option pricing formula does not rely on its precise form. In fact any function of the right shape could be used in its place. Figlewski compares the Black-Scholes formula with an "informationally passive" alternative model, which, following Figlewski, we refer to as the FIG model[1]. The point is that the FIG model is not chosen to provide a best fit, but rather is a simplest attempt at finding a pricing function of approximately the right shape.

There are two distinct usages of the Black-Scholes model. In the first usage a trader calculates the implied volatility from a single option and uses that volatility to calculate the price of a related security. (For each different security the trader wishes to price he might calibrate with a different option.) In the second usage the trader calculates the best-fit implied volatility from

[1]Here FIG can either be taken as an acronym for *flexible implied G* or an abbreviation of the author's surname.

a set of traded options of different strikes, and uses that volatility to give prices for each of a different set of options. The first of these approaches recognizes that market data admits smiles and skews and allows the trader to account for this. However, in doing this the trader is being inconsistent in his use of Black-Scholes. On the one hand he is assuming that volatility is constant (when applying the Black-Scholes pricing function) and on the other he is assuming that different volatilities can be applied in different cases. The second situation suffers no such inconsistency, but then the trader cannot match his model to market data, he can only give a best fit.

Figlewski (2002) tests his alternative model against Black-Scholes in both of these usages. In the first case he uses today's option value to predict the price tomorrow of an option with the same strike and maturity. In the second case he uses today's prices of all the options of a given maturity to calculate a best fit volatility, which is then used to predict tomorrow's prices for those same options.

Figlewski finds that his model provides roughly as good a fit to the data as the Black-Scholes model. In the first case, when the Black-Scholes model is used inconsistently, it tends to outperform the passive alternative, whereas when Black-Scholes is used consistently, the FIG model provides a better fit.

Unfortunately the model FIG admits simple arbitrage. For market parameters (based on the data used both by Figlewski and in this paper) the Figlewski model would give a price ranging from 50 cents to $2 for a put option with zero strike, which must by necessity be worthless.

With this in mind, our paper makes at least four contributions to the literature. Firstly we propose a modified "informationally passive" model

MFIG, satisfying static arbitrage constraints.[2] Secondly, we show how to incorporate a time parameter into both the FIG and MFIG models to produce models FIGT and MFIGT. This allows us to compare options of different maturities as well as different strikes. Thirdly, we show how knowledge of the option price functions given by FIGT and MFIGT can be used to calculate implicit dynamics for the underlying process. Here we use the approach of Dupire (1993,1994). Fourthly, in addition to these theoretical contributions, we test the performance of our model empirically against both the Figlewski and Black-Scholes models.

For this empirical test we use the same dataset as Figlewski (2002), namely traded options on the S&P 500 over the period January 2nd, 1991 to December 29th, 1995.[3] This is so that we can compare our results to his directly.

We find that the modified model gives very similar performance to Figlewski's original model. In general MFIG outperforms the Black-Scholes model in exactly the same situations as the original Figlewski model. More especially if an implied volatility is calculated for each option then Black-Scholes outperforms both the FIG and MFIG models. However if Black-Scholes is used consistently then FIG and MFIG both outperform the Black-

[2] In defence of Figlewski's (2002) original model, although the model misprices a put with with zero strike at $2 and MFIG prices it at zero, over the range of traded options, the differences between the two models are very small. To this extent, the fact that FIG admits static arbitrage can be viewed as a theoretical problem that has little impact in practice. Indeed, we also find many circumstances in which FIG provides a better fit to data than MFIG.

[3] There is good reason to believe that both of the models FIG and MFIG would fit option price data with a symmetric smile, such perhaps as currency option data, better than index option data for which implied volatilities display a skew.

Scholes pricing function.

We find that both FIG and MFIG fit best for low strike options (in-the-money calls and out-of-the-money puts). This is the exact opposite of the behavior reported by Figlewski (2002). We are not able to explain how Figlewski obtains the opposite result from the same dataset. Instead, we give a plausible explanation, using the implied volatility smile and skew of the data, of why the results we report fit with market behavior.

The paper is organized as follows. Static no-arbitrage criteria and the Black and Scholes (1973) option pricing model are given in Section 1. In Section 2, we describe the Figlewski (2002) model and our proposed modified model. These models are extended to allow for maturity dependence in Section 3. Section 4 describes the data used in the empirical testing of the pricing models. Our modified models are tested against the Black and Scholes (1973) pricing formula and the model of Figlewski (2002) in Section 5 and the results reported in terms of RMSE. In the penultimate section we discuss these results and give explanations where appropriate, before we conclude in a final section. The origins of the form of the proposed option pricing functions are given in the Appendix.

1 Static No-Arbitrage Properties

Let C denote the price of a European call option on the stock index level S_t, with strike X, maturity T and riskless rate r. We can write

$$C = C(t, S_t, T, X) = e^{-r(T-t)}\mathbb{E}_t(S_T - X)^+ \tag{1}$$

where t is the current time, S_T the realized value of the index at maturity and expectations are taken with respect to the risk-neutral measure. We assume S has been adjusted for dividends. When we think of a fixed option with given strike and maturity, perhaps when deriving a pricing equation for C, it is usual to think of $C = C_{T,X}(t, S_t)$ as a function of current time and index level. Conversely when we think of the market prices of a family of traded options at a fixed moment in time we should think of $C = C_{t,S_t}(T, X)$ as a function of strike and maturity. The same ideas apply to the price $P = P(t, S_t, T, X) = P_{T,X}(t, S_t) = P_{t,S_t}(T, X)$ of a put option.

There are a number of important properties that option prices must satisfy in order to exclude simple static arbitrages. Merton (1973) derives these properties for stock options. In order to exclude static arbitrages we must have (i)-(iv):

(i) $C_{t,S_t}(T, X)$ is a decreasing, convex function of the strike.

(ii) The current price of a call option with zero strike is equal to the stock price

$$C_{t,S_t}(T, 0) = \lim_{X \downarrow 0} C_{t,S_t}(T, X) = S_t.$$

(iii) The call value is increasing in maturity: for $T \geq \hat{T} \geq t$

$$C_{t,S_t}(T, Xe^{r(T-t)}) \geq C_{t,S_t}(\hat{T}, Xe^{r(\hat{T}-t)}).$$

(iv) Put-call parity holds

$$C_{t,S_t}(T, X) - P_{t,S_t}(T, X) = S_t - Xe^{-r(T-t)}.$$

It is also the case that for any model for which option prices are consistent with (1):

(v) Far out-of-the-money call prices approach zero

$$\lim_{X \uparrow \infty} C_{t,S_t}(T, X) = 0.$$

(vi) At-the-money options have positive time value, for $T > 0$

$$C_{t,S_t}(T, S_t e^{r(T-t)}) > 0.$$

Many of these properties have analogous forms for the call price function $C_{T,X}(t, S_t)$. In particular,

(vii) $C_{T,X}(t, S_t)$ is an increasing, convex function of the asset price.

(viii) Far out-of-the-money call prices approach zero

$$C_{T,X}(t, 0) = \lim_{S_t \downarrow 0} C_{T,X}(t, S_t) = 0.$$

(ix) Far in-the-money call prices approach the value of a long forward contract with the same strike

$$\lim_{S_t \uparrow \infty} \left\{ C_{T,X}(t, S_t) - (S_t - e^{-r(T-t)} X) \right\} = 0.$$

The Black and Scholes (1973) call option pricing formula, which satisfies properties (i)-(ix) is given by

$$C(t, S_t, X, T) = C = S_t N(d_+) - X e^{-r(T-t)} N(d_-) \tag{2}$$

where σ is the volatility of the stock price and

$$d_{\pm} = \frac{\ln(S_t e^{r\tau}/X) \pm (\sigma^2 \tau/2)}{\sigma\sqrt{\tau}}.$$

Here $\tau = T - t$ is the time to maturity. The put price is given via put-call parity in (iv).

Figure 1 plots the Black and Scholes (1973) call pricing formula. The graph on the left plots option prices as a function of S_t and the rightmost graph plots prices as a function of strike X. Parameter values are chosen to be typical of the data analyzed in later sections.

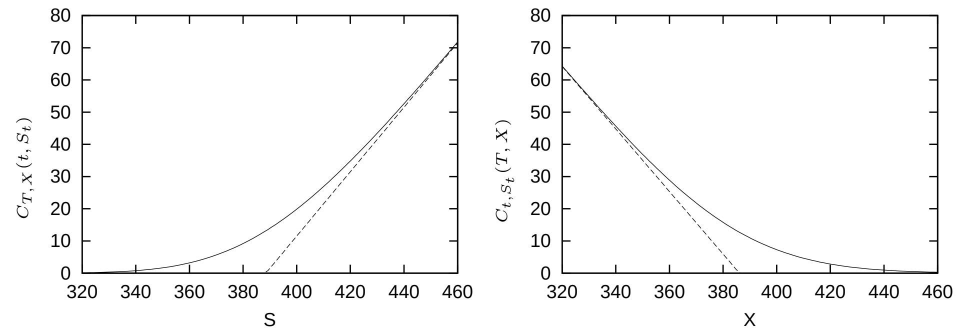

Figure 1: *The Black-Scholes call price. The left graph plots price as a function of S_t whilst the right graph plots price as a function of strike for a fixed value of S_t. In both cases the current time t and maturity T are fixed.*

2 The Figlewski model and Arbitrage-free Modifications

In the previous section we wrote down a minimal list of conditions that an option pricing function must satisfy in order to preclude arbitrage. In this section we describe the FIG model, show that it fails to satisfy some of these conditions and propose a modification MFIG which satisfies all properties (i)-(ix) of Section 1.

Let $f^{\mathcal{G}}_{t,S_t}(T,X)$ denote the time t price of a call on stock index level S_t, with strike X, maturity T, riskless rate r, and parameterized by $\mathcal{G}$. This call pricing function must satisfy (i)-(iii). Suppose $f^{\mathcal{G}}_{t,S_t}(T,X)$ is increasing in $\mathcal{G}$ so that $\mathcal{G}$ plays the role of an implied volatility parameter. Then, given a market call price we can infer $\mathcal{G}$ and substitute this back into the formula to price a related option. For example given today's market price of an option we can infer the implied value of $\mathcal{G}$ and use it to give a price tomorrow for an option with the same strike and maturity. (Of course the value of the index may have changed during this period.)

Figlewski (2002) uses the function

$$FIG^G_{t,S_t}(T,X) = \sqrt{G + \frac{(S_t - Xe^{-r(T-t)})^2}{4}} + \frac{(S_t - Xe^{-r(T-t)})}{2} \qquad (3)$$

which we refer to as the FIG model. Put-call parity in (iv) defines the put price to be

$$FIG^G_{t,S_t}(T,X) - (S_t - Xe^{-r(T-t)}).$$

However, notice that if the strike approaches zero in the FIG model (3),

$$FIG^G_{t,S_t}(T,0) = \lim_{X\downarrow 0} FIG^G_{t,S_t}(T,X) = \sqrt{G + \frac{S_t^2}{4}} + \frac{S_t}{2} > S_t$$

for $G > 0$[4]. Thus this choice of function is inadmissible as a call price function as property (ii) is violated and the FIG model admits arbitrage. In particular, there is a difference between the price of a call option on the stock with strike zero and a unit of the stock itself.[5]

We propose instead to use the modified function

$$MFIG^g_{t,S_t}(T,X) = \sqrt{gS_t + \frac{(S_t - Xe^{-r(T-t)} - g)^2}{4}} + \frac{S_t - Xe^{-r(T-t)} - g}{2} \qquad (4)$$

which we refer to as the MFIG model[6]. Here g plays the role of the implied

[4]Note that if we think instead of the call price as a function of current time and the index level S_t, then property (viii) does not hold for the FIG model. We have

$$FIG^G_{T,X}(t,0) = \lim_{S_t\downarrow 0} FIG^G_{T,X}(t,S_t) > 0.$$

[5]The problem with a model with call prices given by the function FIG is that it is consistent with a price process which can go negative.

[6]The modified flexible implied G model

volatility parameter. For this function

$$MFIG^g_{t,S_t}(T,0) = \lim_{X \downarrow 0} MFIG^g_{t,S_t}(T,X) = S_t$$

and if put prices are given by put-call parity then MFIG satisfies all the necessary conditions for no-arbitrage.

In general it is quite difficult to construct option pricing functions satisfying the no-arbitrage properties (i)-(iii). A motivation and origin for the choice of both of the functions FIG and MFIG is explained in the appendix. These ideas allow us to construct a family of candidate pricing functions. However, as Figlewski is careful to point out, the aim is not to find a best-fit model, but rather to compare the Black-Scholes model against a typical, or rather the simplest, alternative model satisfying no-arbitrage.

3 Time Dependence and Price Dynamics

Unlike the Black and Scholes (1973) pricing formulas, the FIG and MFIG models do not explicitly depend on the option maturity, apart from in the discounting terms.

We can adjust both models to include a maturity dependence. The simplest way to do this is to replace the constant parameters G and g with the functions $(T-t)G$ and $(T-t)g$ which are proportional in time[7]. We get

$$FIGT^G_{t,S_t}(T,X) = \sqrt{G(T-t) + \frac{(S_t - Xe^{-r(T-t)})^2}{4}} + \frac{(S_t - Xe^{-r(T-t)})}{2} \tag{5}$$

[7]More generally we could have used any increasing functions of time to maturity, but our aim is to give the simplest possible extension to the time varying case.

and

$$\begin{aligned} MFIGT^g_{t,S_t}(T,X) &= \sqrt{gS_t(T-t) + \frac{(S_t - Xe^{-r(T-t)} - g(T-t))^2}{4}} \\ &\quad + \frac{S_t - Xe^{-r(T-t)} - g(T-t)}{2} \end{aligned} \tag{6}$$

We refer to (5) and (6) as time modified FIG and MFIG, respectively.

One reason for introducing maturity dependence is to allow us to compare options with different times to expiry. The second and more fundamental reason is to better understand the model.

In his paper Figlewski (2002) states that, unlike the situation in the Black-Scholes model, using his informationally passive alternative requires making no assumptions about the dynamics of the underlying process. However, as we shall see, once maturity has been introduced into the pricing model then the dynamics for the underlying have been specified. Even if maturity is not introduced into the model, then to be consistent the pricing function must have an extension to include maturities, and hence must belong to a severely restricted class of candidate price processes.

In the subsequent analysis we follow Dupire (1993,1994). Under the assumption that the underlying price process is a diffusion, and given European call prices $C_{t,S_t}(T,X)$, then the risk neutral price process for the spot is fully determined. There is a unique diffusion coefficient $a_c(S_u,u)$ such that the index level follows the stochastic differential equation

$$dS_u = rS_u du + a_c(S_u,u)dW_u \tag{7}$$

under the risk neutral probability measure. In fact, the implied dynamics depend on the current call prices, and we can write

$$a_c(x,u) = \left.\sqrt{2\left(\frac{\frac{\partial C_{t,S_t}(T,X)}{\partial T} + rX\frac{\partial C_{t,S_t}(T,X)}{\partial X}}{\frac{\partial^2 C_{t,S_t}(T,X)}{\partial X^2}}\right)}\,\right|_{X=x,T=u} \tag{8}$$

where $C_{t,S_t}(T,X)$ is the call price function, thought of as a function of strike.

The dynamics for the index level under the time modified FIG model can be calculated using (8) and (5) as the call pricing function. The index follows (7) with

$$a_c(S_u,u) = \sqrt{\frac{4}{u-t}\left(G(u-t)e^{2r(u-t)} + \frac{(S_t e^{r(u-t)} - S_u)^2}{4}\right)} \quad (9)$$

Similarly, the index level under the time modified MFIG model follows (7) with

$$a_c(S_u,u) = \sqrt{\frac{2D^2 e^{2r(u-t)}(S_t + S_u e^{-r(u-t)} + g(u-t) - 2D)}{S_t(u-t)}} \quad (10)$$

where $D^2 = gS_t(u-t) + \frac{(S_t - S_u e^{-r(u-t)} - g(u-t))^2}{4}$.

Notice first the contrast between the diffusion coefficients driving the index level under these two models and the Samuelson (1965) model used for the Black and Scholes (1973) option pricing equation. Under the Samuelson model,

$$dS_u = rS_u du + \sigma S_u dW_u$$

so $a_c(S_u,u) = \sigma S_u$ with σ constant. In both time modified models, the diffusion coefficients (9) and (10) depend upon the constant parameters G and g, current index level S_u, but also on current time, and the initial index level S_t. Hence, although the option pricing functions (5) and (6) are not too complicated, the implied index dynamics consistent with these functions are much more complicated than the lognormal model of Samuelson (1965) and Black and Scholes (1973).

Now compare the dynamics for the index under the two time modified models FIG and MFIG. Notice that for the FIGT model,

$$a_c(0,u) > 0$$

whereas for the time modified MFIG model,

$$a_c(0, u) = 0.$$

In particular, in the time extended FIG model, when the index hits zero its diffusion coefficient is non-zero and the price process can and does go negative. Conversely, in the modified model, MFIGT, when the index first hits zero, the diffusion coefficient is also zero and the process stops. This explains why the FIG model gives positive value to put options with zero strike, whereas MFIG correctly gives a zero value to these options.

4 The Data

The data used in this study is daily data on S&P 500 index options taken from the Berkeley Options Database. Option prices correspond to the average of the last bid and ask quotes reported before 3:00PM CST. We also use values of the S&P500 index, dividend payout on the S&P 500 over the remaining option life and riskless interest rates. The data runs from January 2nd, 1991 to December 29, 1995.

The data is the same as used by Figlewski (2002). This allows us to make a direct comparison with his results.

We construct a dividend adjusted index value by subtracting the present value of the dividends over the remaining option life from the raw series. This dividend adjusted series is used in place of the raw series. The interest rate is LIBOR obtained from the British Bankers Association, interpolated between adjacent months. We disregarded any observations where option prices violated arbitrage bounds or where implied volatilities were unable to be calculated. There were also a very small number of observations with