Optimizing a Flywheel Profile

Introduction

The radial stress component in an axially symmetric and homogeneous flywheel of constant thickness exhibits a sharp peak near the inner radius. From there, it decreases monotonously until it reaches zero at the flywheel's outer rim; see Figure 1. The uneven stress distribution—apparent also for the azimuthal component—reveals a design that does not make optimal use of the material available.

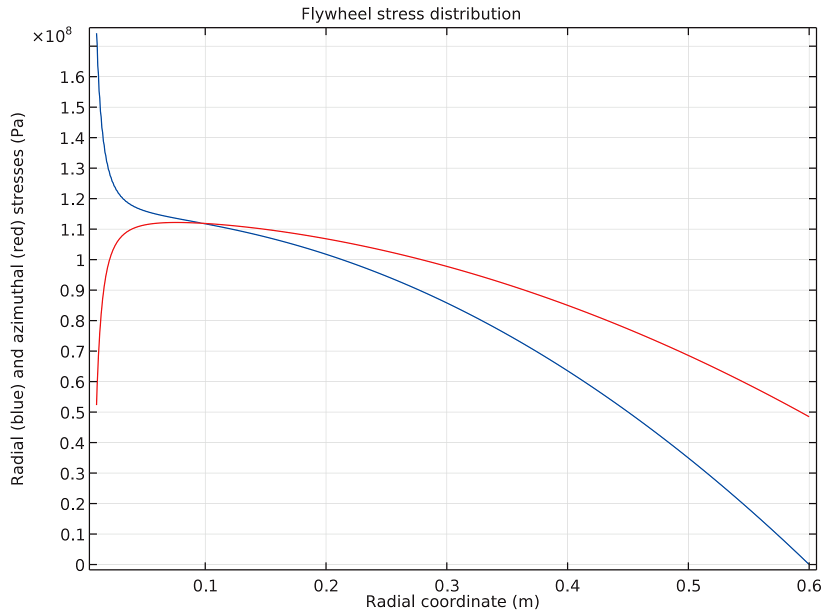

Figure 1: Radial (blue) and azimuthal (red) stress components in a homogeneous flywheel of constant thickness.

This model solves the problem of finding the thickness profile that results in a radial stress distribution that is as even as possible for given values of the flywheel's mass and moment of inertia. The model was inspired by Ref. 1.

Model Definition

Before describing the optimization problem, this section derives the dynamical equations, which you implement using a General Form PDE interface in 1D.

STRESSES IN A ROTATING FLYWHEEL

In a rotating flywheel, stresses due to the flywheel's weight are typically very small compared to dynamically induced stresses; therefore this model neglects gravitational stress. Expressed in terms of $U \equiv u/r$, where u is the local radial displacement (m) and r is the radial coordinate, the stress components along the radial and azimuthal directions in a rotationally symmetric disk made of a homogeneous, isotropic, and elastic material with Young's modulus E (N/m^2) and Poisson's ratio ν read:

$$\begin{aligned}\sigma_r &= \frac{E}{1-\nu^2}\left[r\frac{dU}{dr} + (1+\nu)U\right] \\ \sigma_\varphi &= \frac{E}{1-\nu^2}\left[\nu r\frac{dU}{dr} + (1+\nu)U\right]\end{aligned} \qquad (1)$$

Inserting these expressions in the equation of motion for an infinitesimal mass element, results in the second-order ordinary differential equation (ODE)

$$-r^2\frac{d^2U}{dr^2} - (3+\Phi)r\frac{dU}{dr} + (1-(1+\nu)\Phi)U = \frac{1-\nu^2}{E}\rho\omega^2 r^2 \qquad r_0 < r < r_1 \qquad (2)$$

valid for a centrally bored flywheel with inner radius r_0 (m) and outer radius r_1 (m) rotating with the angular velocity ω (rad/s). In this equation, the flywheel's thickness, H, which can be a function of r, enters through the dimensionless function

$$\Phi \equiv \frac{r}{H}\frac{dH}{dr} = r\frac{d}{dr}\log\left(\frac{H}{H_0}\right)$$

At the inner radius the displacement is zero and at the outer radius the radial stress component vanishes, which corresponds to the following boundary conditions:

$$U\big|_{r=r_0} = 0, \qquad r\frac{dU}{dr} + (1+\nu)U\big|_{r=r_1} = 0 \qquad (3)$$

Given the function Φ, Equation 2 combined with Equation 3 forms a well-posed ODE problem. With the solution $U = U(r)$ at hand, you can determine the stress components through the Equation 1.

THE OPTIMIZATION PROBLEM

For the special case of constant flywheel thickness, $H(r) = H_0$, the function Φ is identically zero. As Figure 1 shows and you verify later, this shape results in an uneven stress distribution, with a maximum for σ_r at $r = r_0$.

This model concerns optimizing the flywheel's profile to obtain a radial stress distribution that is as even as possible under the design requirements of specified flywheel mass and moment of inertia. To formulate the task in mathematical terms, tentatively introduce the objective function

$$Q_{\text{stress}}[H] = \int_{r_0}^{r_1} \frac{(\sigma_r - \sigma_{r,\text{ mean}})^2}{\sigma_0^{\ 2}} dr \qquad (4)$$

where $\sigma_{r,\text{mean}}$ denotes the average radial stress value along the flywheel's radial extension, and σ_0 is a normalization constant. The latter is introduced to make the integrand dimensionless and its value is chosen to be roughly an order of magnitude smaller than σ_r to give Q_{stress} a suitable magnitude. The optimization problem is then to find the shape $H = H(r)$ that minimizes Q_{stress} under the additional constraints

$$m \equiv 2\pi\rho \int_{r_0}^{r_1} H r dr = m_0$$

and

$$I \equiv \pi\rho \int_{r_0}^{r_1} H r^3 dr = I_0$$

where m_0 and I_0 are the desired mass and moment of inertia, respectively.

However, Equation 4 alone does not give a reasonable result; suppressing profiles where dH/dr is not smooth requires a second term in the objective function:

$$Q_{\text{smoothness}}[H] = A \int_{r_0}^{r_1} \left(\frac{dH}{dr}\right)^2 dr$$

Here A is a normalization constant to be chosen such that $Q_{\text{smoothness}}$ and Q_{stress} are comparable in magnitude; as long as this condition is satisfied, the model is fairly insensitive to the value of A.

MODEL DATA

Table 1 gives the input data for the model. As the initial design, take a flywheel of constant thickness H_0. The material properties correspond to those of steel.

TABLE 1: MODEL DATA

PROPERTY	VALUE	DESCRIPTION
r_0	0.01 m	Inner flywheel radius
r_1	0.60 m	Outer flywheel radius
H_0	0.03 m	Initial flywheel thickness
E	$2.1 \cdot 10^{11}$ N/m^2	Young's modulus
ν	0.3	Poisson's parameter
ρ	7800 kg/m^3	Density
ω	$2\pi \cdot 50$ rad/s	Angular velocity
σ_0	10^7 Pa	Normalization constant, stress term
A	1	Normalization constant, smoothness term

Results and Discussion

Figure 2 shows the optimized flywheel profile (black lines) with that of the original flat flywheel of the same mass and moment of inertia included for comparison (green lines).

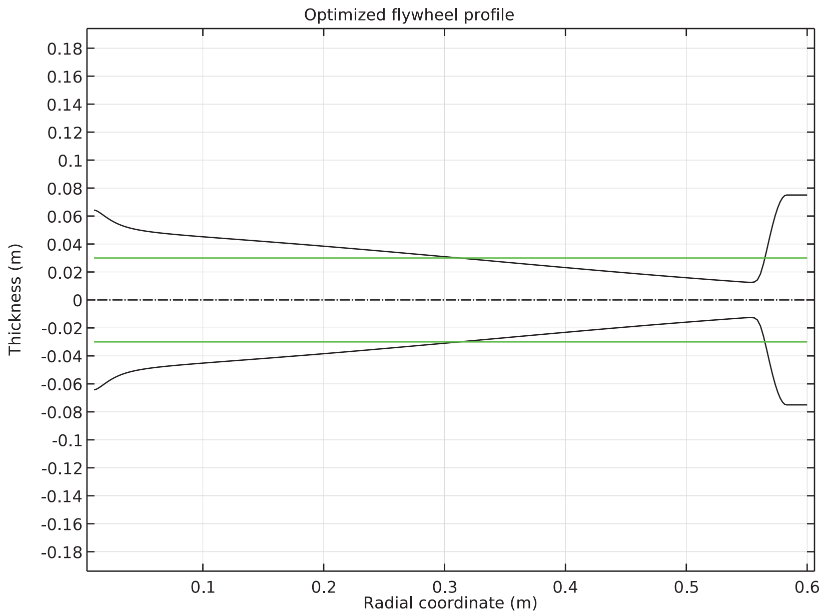

Figure 2: Optimized thickness profile.

Figure 3 displays the radial and azimuthal stress components for both the initial and the optimized flywheel profiles. In the optimized flywheel, the stress components are almost equal and nearly constant for most of the radial cross-section. The maximal stress, which occurs in the radial direction at the inner radius, is roughly 102 MPa for