INCOMPLETE ELLIPTIC INTEGRALS IN RAMANUJAN'S LOST NOTEBOOK

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Dedicated with appreciation and thanks to Richard Askey on his 65^{th} birthday

. On pages 51–53 of his lost notebook, S. Ramanujan expressed several integrals of products of Dedekind eta-functions in terms of incomplete elliptic integrals of the first kind. In this paper, we prove these identities using only results found in Ramanujan's notebooks. We then construct several new elliptic integrals of this type using modular identities associated with certain "Hauptmoduls."

1. Introduction

On pages 51–53 in his lost notebook [17], Ramanujan recorded several identities involving integrals of theta-functions and incomplete elliptic integrals of the first kind. We offer here one typical example, proved in Theorem 7.5 below. Let (in Ramanujan's notation) $f(-q) = (q;q)_\infty$. (Detailed notation is given in Section 2. The function f is essentially the Dedekind eta-function; see (2.4).) Let

$$v := v(q) := q\frac{f^3(-q)f^3(-q^{15})}{f^3(-q^3)f^3(-q^5)}. \tag{1.1}$$

Then

(1.2)

$$\int_0^q f(-t)f(-t^3)f(-t^5)f(-t^{15})dt = \frac{1}{5}\int_{2\tan^{-1}\left(\frac{1}{\sqrt{5}}\sqrt{\frac{1-11v-v^2}{1+v-v^2}}\right)}^{2\tan^{-1}(1/\sqrt{5})} \frac{d\varphi}{\sqrt{1-\frac{9}{25}\sin^2\varphi}}.$$

The reader will immediately realize that these are rather uncommon integrals. Indeed, we have never seen identities like (1.2) in the literature.

In a wonderful paper [13], all of these integral identities were proved by S. Raghavan and S. S. Rangachari. However, in almost all of their proofs, they used results with which Ramanujan would have been unfamiliar. In particular, they relied heavily on results from the theory of modular forms, evidently not known to Ramanujan. For example, for four identities, including (1.2), Raghavan and Rangachari appealed to differential equations

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satisfied by certain quotients of eta-functions, such as (1.1), which can be found in R. Fricke's text [9].

In an effort to discern Ramanujan's methods and to better understand the origins of identities like (1.2), the present authors have devised proofs independent of the theory of modular forms and other ideas with which Ramanujan would have been unfamiliar. In particular, we have relied exclusively on results found in his ordinary notebooks [15] and his lost notebook [17]. It should be emphasized that at the time of the publication of Raghavan and Rangachari's paper [13] a decade ago, many of these results had not yet been proved. Particularly troublesome for us were the aforementioned four differential equations for quotients of eta-functions. To prove these, we used identities for Eisenstein series found in Chapter 21 of Ramanujan's second notebook and several eta-function identities scattered among the unorganized pages of his second notebook [2, Chap. 25]. We have also utilized several results in the lost notebook found on pages in close proximity to the elliptic integral identities.

The authors owe a huge debt to Raghavan and Rangachari's paper [13]. In many cases, we have incorporated large portions of their proofs, while in other instances we have employed different lines of attack. This paper could have been made shorter by referring to their paper for large portions of certain proofs, but considerable readability would have been lost in doing so.

In Section 3, we prove two identities for integrals of theta-functions of forms unlike (1.2). The first proof is virtually the same as that given by Raghavan and Rangachari, while the latter proof is completely different. In Sections 4–6, we prove several integral identities associated with modular equations of degree 5. Here some transformations of incomplete elliptic integrals due to J. Landen and Ramanujan play key roles. In Section 7, several identities of order 15 are established. Here two of the aforementioned differential equations are crucial. Differential equations are also central in Sections 8 and 9, where identities of orders 14 and 35, respectively, are proved.

Since differential equations for quotients of eta-functions are of such paramount importance in proving identities akin to (1.2), we have systematically derived several new differential equations for eta-function quotients in Section 10. We have used two of these new differential equations to derive two new formulas in the spirit of (1.2). In Section 10, we also point out the connection of such integrals with elliptic curves. We plan to return to these matters in a future paper.

2. Preliminary Results

As usual, set, for each nonnegative integer n,

$$(a;q)_n = \prod_{k=0}^{n-1}(1-aq^k)$$

and

$$(a;q)_\infty = \lim_{n\to\infty}(a;q)_n, \qquad |q|<1.$$

Ramanujan's general theta-function $f(a,b)$ is defined by

$$f(a,b) = \sum_{n=-\infty}^{\infty} a^{n(n+1)/2}b^{n(n-1)/2}, \qquad |ab|<1.$$

Theta-functions satisfy the very important and useful Jacobi triple product identity [1, p. 35, Entry 19],

$$f(a,b) = (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty. \tag{2.1}$$

The most important special cases are given by

$$\varphi(q) := f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2} = \frac{(-q;q^2)_\infty(q^2;q^2)_\infty}{(-q^2;q^2)_\infty(q;q^2)_\infty}, \tag{2.2}$$

$$\psi(q) := f(q,q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}, \tag{2.3}$$

and

$$\begin{aligned} f(-q) := f(-q,-q^2) &= \sum_{n=-\infty}^{\infty}(-1)^n q^{n(3n-1)/2} = (q;q)_\infty \\ &=: e^{-2\pi iz/24}\eta(z), \qquad q=e^{2\pi iz}, \qquad \operatorname{Im} z>0. \end{aligned} \tag{2.4}$$

The product representations in (2.2)–(2.4) are instances of the Jacobi triple product identity (2.1). The function $\eta(z)$, defined in (2.4), is the Dedekind eta-function. It has the transfomation formula

$$\eta(-1/z) = \sqrt{z/i}\,\eta(z). \tag{2.5}$$

The functions φ, ψ, and f in (2.2)–(2.4) can be expressed in terms of the modulus k and the hypergeometric function $z := {}_2F_1(\frac12,\frac12;1;k^2)$. For a catalogue of formulas of this type, see [1, pp. 122–124]. We will need two such formulas in the sequel. If $\alpha = k^2$ and

$$q = \exp\left(\frac{{}_2F_1(\frac12,\frac12;1;1-\alpha)}{{}_2F_1(\frac12,\frac12;1;\alpha)}\right),$$

then

$$\psi(-q) = \sqrt{\tfrac12 z}\,\{\alpha(1-\alpha)/q\}^{1/8} \tag{2.6}$$

and

$$f(-q^2) = \sqrt{z}2^{-1/3}\left\{\alpha(1-\alpha)/q\right\}^{1/12}. \tag{2.7}$$

The Eisenstein series $P(q), Q(q)$, and $R(q)$ are defined by

$$P(q) := 1 - 24\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n}, \tag{2.8}$$

$$= 1 + 240\sum_{n=1}^{\infty}\frac{n^3q^n}{1-q^n}, \tag{2.9}$$

and

$$R(q) := 1 - 504\sum_{n=1}^{\infty}\frac{n^5q^n}{1-q^n}. \tag{2.10}$$

(This is the notation used by Ramanujan in his lost notebook and paper [14], [16, pp. 136–162], but in his ordinary notebooks, P, Q, and R are replaced by L, M, and N, respectively.)

The Rogers–Ramanujan continued fraction $u(q)$ is defined by

$$u := u(q) := \frac{q^{1/5}}{1} \, \frac{}{+} \, \frac{q}{1} \, \frac{}{+} \, \frac{q^2}{1} \, \frac{}{+} \, \frac{q^3}{1} \, \frac{}{+} \cdots, \qquad |q| < 1. \tag{2.11}$$

With $f(-q)$ defined by (2.4), two of the most important properties of $u(q)$ are given by [1, p. 267, eqs. (11.5), (11.6)]

$$\frac{1}{u(q)} - 1 - u(q) = \frac{f(-q^{1/5})}{q^{1/5}f(-q^5)} \tag{2.12}$$

and

$$\frac{1}{u^5(q)} - 11 - u^5(q) = \frac{f^6(-q)}{qf^6(-q^5)}. \tag{2.13}$$

A common generalization of (2.12) and (2.13) was recorded by Ramanujan in his lost notebook and proved by S. H. Son [18]. Lastly, it can be shown that, with the use of the Rogers–Ramanujan identities [1, p. 79],

$$u(q) = q^{1/5}\frac{(q;q^5)_\infty(q^4;q^5)_\infty}{(q^2;q^5)_\infty(q^3;q^5)_\infty}. \tag{2.14}$$

3. Two Simpler Integrals

Theorem 3.1 (p. 51). *Let $P(q), Q(q)$, and $R(q)$ be the Eisenstein series defined by (2.8)–(2.10). Then*

$$\int_{e^{-2\pi}}^{q}\sqrt{Q(t)}\frac{dt}{t} = \log\left(\frac{Q^{3/2}(q)-R(q)}{Q^{3/2}(q)+R(q)}\right).$$

Proof. Following Ramanujan's suggestion, let $z = R^2(t)/Q^3(t)$. Then

$$\frac{1}{z}\frac{dz}{dq} = \frac{2}{R}\frac{dR}{dq} - \frac{3}{Q}\frac{dQ}{dq}. \tag{3.1}$$

Using Ramanujan's differential equations [14, eq. (30)], [17, p. 142], [1, p. 330]

$$q\frac{dR}{dq} = \frac{PR - Q^2}{2} \qquad \text{and} \qquad q\frac{dQ}{dq} = \frac{PQ - R}{3},$$

in (3.1), we find that

$$\frac{q}{z}\frac{dz}{dq} = \frac{R^2 - Q^3}{RQ}. \tag{3.2}$$

Hence, by (3.2),

$$\begin{aligned} q\frac{d}{dq}\log\left(\frac{Q^{3/2} - R}{Q^{3/2} + R}\right) =& q\frac{d}{dq}\log\left(\frac{1 - \sqrt{z}}{1 + \sqrt{z}}\right) \\ =& q\frac{d}{dz}\log\left(\frac{1 - \sqrt{z}}{1 + \sqrt{z}}\right)\frac{dz}{dq} \\ =& \frac{1}{\sqrt{z}(z-1)}q\frac{dz}{dq} \\ =& \sqrt{Q}. \end{aligned}$$

It follows that

$$\begin{aligned} \int_{e^{-2\pi}}^{q} \sqrt{Q(t)}\frac{dt}{t} &= \int_{e^{-2\pi}}^{q} \frac{d}{dt}\log\left(\frac{Q^{3/2} - R}{Q^{3/2} + R}\right) dt \\ &= \log\left(\frac{Q^{3/2}(q) - R(q)}{Q^{3/2}(q) + R(q)}\right) - \log\left(\frac{Q^{3/2}(e^{-2\pi}) - R(e^{-2\pi})}{Q^{3/2}(e^{-2\pi}) + R(e^{-2\pi})}\right). \end{aligned}$$

But it is well-known that $R(e^{-2\pi}) = 0$ [8, p. 88], and so Theorem 3.1 follows. □

Theorem 3.2 (p. 53)**.** *Let $u(q)$ denote the Rogers–Ramanujan continued fraction, defined by (2.11), and set $v = u(q^2)$. Recall that $\psi(q)$ is defined by (2.3). Then*

$$\frac{8}{5}\int \frac{\psi^5(q)}{\psi(q^5)}\frac{dq}{q} = \log(u^2v^3) + \sqrt{5}\log\left(\frac{1 + (\sqrt{5} - 2)uv^2}{1 - (\sqrt{5} + 2)uv^2}\right). \tag{3.3}$$

Proof. Let $k := k(q) := uv^2$. Then from page 53 of Ramanujan's lost notebook [17], or from page 326 of his second notebook [3, pp. 12–13],

$$u^5 = k\left(\frac{1-k}{1+k}\right)^2 \qquad \text{and} \qquad v^5 = k^2\left(\frac{1+k}{1-k}\right). \tag{3.4}$$

(See also S.–Y. Kang's paper [11].) It follows that

$$\log(u^2v^3) = \frac{1}{5}\log\left(k^8\frac{1-k}{1+k}\right). \tag{3.5}$$

If we set $\epsilon = (\sqrt{5}+1)/2$, we readily find that $\epsilon^3 = \sqrt{5}+2$ and $\epsilon^{-3} = \sqrt{5}-2$. Then, with the use of (3.5), we see that (3.3) is equivalent to the equality

$$\frac{8}{5}\int \frac{\psi^5(q)}{\psi(q^5)}\frac{dq}{q} = \frac{1}{5}\log\left(k^8\frac{1-k}{1+k}\right) + \sqrt{5}\log\left(\frac{1+\epsilon^{-3}k}{1-\epsilon^3 k}\right). \tag{3.6}$$

Now from Entry 9(vi) in Chapter 19 of Ramanujan's second notebook [1, p. 258],

$$\frac{\psi^5(q)}{\psi(q^5)} = 25q^2\psi(q)\psi^3(q^5) + 1 - 5q\frac{d}{dq}\log\frac{f(q^2,q^3)}{f(q,q^4)}. \tag{3.7}$$

By the Jacobi triple product identity (2.1),

$$\begin{aligned}\frac{f(q^2,q^3)}{f(q,q^4)} =& \frac{(-q^2;q^5)_\infty(-q^3;q^5)_\infty}{(-q;q^5)_\infty(-q^4;q^5)_\infty}\\ =& \frac{(q;q^5)_\infty(q^4;q^5)_\infty(q^4;q^{10})_\infty(q^6;q^{10})_\infty}{(q^2;q^5)_\infty(q^3;q^5)_\infty(q^2;q^{10})_\infty(q^8;q^{10})_\infty}\\ =& q^{1/5}\frac{u(q)}{v(q)},\end{aligned} \tag{3.8}$$

by (2.14). Using (3.8) in (3.7), we find that

$$\begin{aligned}\frac{8}{5}\int \frac{\psi^5(q)}{\psi(q^5)}\frac{dq}{q} =& 40\int q\psi(q)\psi^3(q^5)dq + \int \frac{8}{5q}dq - 8\int \frac{d}{dq}\log\left(q^{1/5}u/v\right)dq\\ =& 40\int q\psi(q)\psi^3(q^5)dq - 8\log(u/v)\\ =& 40\int q\psi(q)\psi^3(q^5)dq + \frac{8}{5}\log k - \frac{24}{5}\log\frac{1-k}{1+k},\end{aligned} \tag{3.9}$$

where (3.4) has been employed. Comparing (3.9) with (3.6), we now see that it suffices to prove that

$$8\int q\psi(q)\psi^3(q^5)dq = \log\frac{1-k}{1+k} + \frac{1}{\sqrt{5}}\log\left(\frac{1+\epsilon^{-3}k}{1-\epsilon^3 k}\right). \tag{3.10}$$

Upon differentiation of both sides of (3.10) and simplification, we find that (3.10) is equivalent to

$$q\psi(q)\psi^3(q^5) = \frac{k(q)k'(q)}{(1-k^2(q))(1-4k(q)-k^2(q))}. \tag{3.11}$$

We now prove (3.11). By (3.4) again,

$$\frac{v}{u^2} = \frac{1+k}{1-k}. \tag{3.12}$$

Taking the logarithmic derivative of both sides of (3.12), we find that

$$\frac{k'(q)}{1-k^2(q)} = \frac{1}{2}\frac{v'(q)}{v(q)} - \frac{u'(q)}{u(q)}. \tag{3.13}$$

By the logarithmic differentiation of (2.14),

$$\frac{u'(q)}{u(q)} = \frac{1}{5q} - \sum_{n=1}^{\infty} \left(\frac{n}{5}\right) \frac{nq^{n-1}}{1-q^n}$$

and

$$\frac{v'(q)}{v(q)} = 2\left(\frac{1}{5q} - \sum_{n=1}^{\infty} \left(\frac{n}{5}\right) \frac{nq^{2n-1}}{1-q^{2n}}\right),$$

where $\left(\frac{n}{5}\right)$ denotes the Legendre symbol. Using these derivatives in (3.13), we see that

$$\frac{k'(q)}{1-k^2(q)} = \sum_{n=1}^{\infty} \left(\frac{n}{5}\right) \frac{nq^{n-1}}{1-q^{2n}}. \tag{3.14}$$

However, from Entry 8(i) in Chapter 19 of Ramanujan's second notebook [1, p. 249],

$$\sum_{n=1}^{\infty} \left(\frac{n}{5}\right) \frac{nq^n}{1-q^{2n}} = q\psi^3(q)\psi(q^5) - 5q^2\psi(q)\psi^3(q^5),$$

so that, by (3.14),

$$\frac{k'(q)}{1-k^2(q)} = \psi^3(q)\psi(q^5) - 5q\psi(q)\psi^3(q^5). \tag{3.15}$$

From page 56 in Ramanujan's lost notebook [17],

$$\frac{\psi^2(q)}{q\psi^2(q^5)} = \frac{1-k^2(q)}{k(q)} + 1, \tag{3.16}$$

which has been proved by Kang [11, Thm. 4.2]. Putting (3.16) in (3.15), we deduce that

$$\frac{k'(q)}{1-k^2(q)} = \left(\frac{1-k^2(q)}{k(q)} - 4\right) q\psi(q)\psi^3(q^5). \tag{3.17}$$

It is easily seen that (3.17) is equivalent to (3.11), and so the proof of (3.3) is complete. □

4. Elliptic Integrals of Order 5 (I)

Theorem 4.1 (p. 52)**.** *With $f(-q), \psi(q)$, and $u(q)$ defined by (2.4), (2.3), and (2.11), respectively, and with $\epsilon = (\sqrt{5}+1)/2$,*

$$5^{3/4}\int_0^q \frac{f^2(-t)f^2(-t^5)}{\sqrt{t}}dt = 2\int_{\cos^{-1}\left((\epsilon u)^{5/2}\right)}^{\pi/2} \frac{d\varphi}{\sqrt{1-\epsilon^{-5}5^{-3/2}\sin^2\varphi}} \tag{4.1}$$

$$= \int_0^{2\tan^{-1}\left(5^{3/4}\sqrt{q}f^3(-q^5)/f^3(-q)\right)} \frac{d\varphi}{\sqrt{1-\epsilon^{-5}5^{-3/2}\sin^2\varphi}} \tag{4.2}$$

$$= \sqrt{5}\int_0^{2\tan^{-1}\left(5^{1/4}\sqrt{q}\psi(q^5)/\psi(q)\right)} \frac{d\varphi}{\sqrt{1-\epsilon 5^{-1/2}\sin^2\varphi}}. \tag{4.3}$$

To prove (4.1), we need the following lemma.

Lemma 4.2. *Let $u(q)$ be defined by (2.11). Then*

$$u'(q) = \frac{u(q)}{5q}\frac{f^5(-q)}{f(-q^5)}.$$

Proof. By (2.14) and the Jacobi triple product identity (2.1),

$$u(q) = q^{1/5}\frac{(q;q^5)_\infty(q^4;q^5)_\infty}{(q^2;q^5)_\infty(q^3;q^5)_\infty} = q^{1/5}\frac{f(-q,-q^4)}{f(-q^2,-q^3)}.$$

By logarithmic differentiation and the use of Entry 9(v) in Chapter 19 of Ramanujan's second notebook [1, p. 258],

$$\begin{aligned}\frac{u'(q)}{u(q)} &= \frac{1}{5q} + \frac{d}{dq}\log\frac{f(-q,-q^4)}{f(-q^2,-q^3)} \\ &= \frac{1}{5q} + \frac{1}{5q}\left(-1+\frac{f^5(-q)}{f(-q^5)}\right) = \frac{1}{5q}\frac{f^5(-q)}{f(-q^5)},\end{aligned}$$

which completes the proof. □

Proof of (4.1). Let

$$\cos^2\varphi = \epsilon^5 u^5(t). \tag{4.4}$$

If $t=0$, then $\varphi = \pi/2$; if $t=q$, then $\varphi = \cos^{-1}\left((\epsilon u)^{5/2}\right)$. Upon differentiation and the use of Lemma 4.2,

$$\begin{aligned}2\cos\varphi(-\sin\varphi)\frac{d\varphi}{dt} &= 5\epsilon^5 u^4(t)u'(t) \\ &= \epsilon^5\frac{u^5(t)}{t}\frac{f^5(-t)}{f(-t^5)} = \cos^2\varphi\frac{f^5(-t)}{tf(-t^5)}.\end{aligned} \tag{4.5}$$

Hence, by (4.5), (2.13), and (4.4),

$$
\begin{aligned}
&5^{3/4}\int_0^q \frac{f^2(-t)f^2(-t^5)}{\sqrt{t}}dt\\
=&5^{3/4}\int_{\pi/2}^{\cos^{-1}\left((\epsilon u)^{5/2}\right)} \frac{f^2(-t)f^2(-t^5)}{\sqrt{t}}\frac{-2tf(-t^5)}{f^5(-t)}\frac{\sin\varphi}{\cos\varphi}d\varphi\\
=&2\cdot 5^{3/4}\int_{\cos^{-1}\left((\epsilon u)^{5/2}\right)}^{\pi/2} \sqrt{t}\frac{f^3(-t^5)}{f^3(-t)}\frac{\sin\varphi}{\cos\varphi}d\varphi\\
=&2\cdot 5^{3/4}\int_{\cos^{-1}\left((\epsilon u)^{5/2}\right)}^{\pi/2} \frac{1}{\sqrt{1/u^5(t)-11-u^5(t)}}\frac{\sin\varphi}{\cos\varphi}d\varphi\\
=&2\cdot 5^{3/4}\int_{\cos^{-1}\left((\epsilon u)^{5/2}\right)}^{\pi/2} \frac{\sin\varphi}{\sqrt{\epsilon^5-11\cos^2\varphi-\epsilon^{-5}\cos^4\varphi}}d\varphi. \qquad (4.6)
\end{aligned}
$$

Since $\epsilon^{\pm 5}=(5\sqrt{5}\pm 11)/2$,

$$
\begin{aligned}
\epsilon^5-11\cos^2\varphi-\epsilon^{-5}\cos^4\varphi =&\epsilon^5-11(1-\sin^2\varphi)-\epsilon^{-5}\cos^4\varphi\\
=&\epsilon^{-5}+11\sin^2\varphi-\epsilon^{-5}\cos^4\varphi\\
=&\epsilon^{-5}(1-\cos^2\varphi)(1+\cos^2\varphi)+11\sin^2\varphi\\
=&\epsilon^{-5}\sin^2\varphi(2-\sin^2\varphi)+11\sin^2\varphi\\
=&\sin^2\varphi(2\epsilon^{-5}+11-\epsilon^{-5}\sin^2\varphi)\\
=&\sin^2\varphi(5\sqrt{5}-\epsilon^{-5}\sin^2\varphi)\\
=&5\sqrt{5}\sin^2\varphi(1-\epsilon^{-5}5^{-3/2}\sin^2\varphi).
\end{aligned}
$$

Thus, from (4.6),

$$
5^{3/4}\int_0^q \frac{f^2(-t)f^2(-t^5)}{\sqrt{t}}dt = 2\int_{\cos^{-1}\left((\epsilon u)^{5/2}\right)}^{\pi/2}\frac{d\varphi}{\sqrt{1-\epsilon^{-5}5^{-3/2}\sin^2\varphi}},
$$

which is (4.1). □

To prove (4.2), we need two transformations for incomplete elliptic integrals found in Chapter 17 of Ramanujan's second notebook [1, pp. 105–106, Entries 7(ii), (vi)].

Lemma 4.3. *If* $\tan\gamma=\sqrt{1-x}\ \tan\alpha$, *then*

$$
\int_0^\alpha \frac{d\varphi}{\sqrt{1-x\sin^2\varphi}}=\int_0^\gamma\frac{d\varphi}{\sqrt{1-x\cos^2\varphi}}. \qquad (4.7)
$$

If $\cot\alpha\ \tan(\beta/2)=\sqrt{1-x\sin^2\alpha}$, *then*

$$
2\int_0^\alpha \frac{d\varphi}{\sqrt{1-x\sin^2\varphi}}=\int_0^\beta\frac{d\varphi}{\sqrt{1-x\sin^2\varphi}}. \qquad (4.8)
$$

Proof of (4.2). In (4.7), replace φ by $\pi/2-\varphi$ and combine the result with (4.8) to deduce that

$$\int_0^{\beta}\frac{d\varphi}{\sqrt{1-x\sin^2\varphi}}=2\int_{\pi/2-\gamma}^{\pi/2}\frac{d\varphi}{\sqrt{1-x\sin^2\varphi}}, \tag{4.9}$$

provided that
(i) $\cot\alpha\ \tan(\beta/2)=\sqrt{1-x\sin^2\alpha}$,
(ii) $\tan\gamma=\sqrt{1-x}\ \tan\alpha$.

Examining (4.1) and (4.2), we see that we want to set $x=\epsilon^{-5}5^{-3/2}$ and $\gamma=\frac{\pi}{2}-\cos^{-1}\left((\epsilon u)^{5/2}\right)$. We also see that, to prove (4.2), we will need to show that (i) and (ii) imply that

$$\beta=2\tan^{-1}\left(5^{3/4}\sqrt{q}f^3(-q^5)/f^3(-q)\right). \tag{4.10}$$

Since $\epsilon^{\pm5}=(5\sqrt{5}\pm11)/2$, a short calculation gives

$$1-\epsilon^{-5}5^{-3/2}=\epsilon^{5}5^{-3/2}.$$

Thus, from (ii) and elementary trigonometry,

$$\begin{aligned}\tan\alpha=&\frac{1}{\sqrt{1-\epsilon^{-5}5^{-3/2}}}\cot\left(\cos^{-1}(\epsilon u)^{5/2}\right)\\ =&\epsilon^{-5/2}5^{3/4}\frac{(\epsilon u)^{5/2}}{\sqrt{1-(\epsilon u)^5}}=\frac{5^{3/4}u^{5/2}}{\sqrt{1-(\epsilon u)^5}}.\end{aligned} \tag{4.11}$$

Thus, by (i),

$$\tan(\beta/2)=\sqrt{1-\epsilon^{-5}5^{-3/2}\sin^2\alpha}\frac{5^{3/4}u^{5/2}}{\sqrt{1-(\epsilon u)^5}}. \tag{4.12}$$

From (4.11) and elementary trigonometry,

$$x\ \sin^2\alpha=\frac{\epsilon^{-5}u^5}{1+\epsilon^{-5}u^5}.$$

Using this in (4.12), we deduce that

$$\begin{aligned}\tan(\beta/2)=&\sqrt{1-\frac{\epsilon^{-5}u^5}{1+\epsilon^{-5}u^5}}\frac{5^{3/4}u^{5/2}}{\sqrt{1-(\epsilon u)^5}}\\ =&\frac{5^{3/4}u^{5/2}}{\sqrt{(1+\epsilon^{-5}u^5)(1-\epsilon^5u^5)}}\\ =&\frac{5^{3/4}u^{5/2}}{\sqrt{1-11u^5-u^{10}}}\\ =&\frac{5^{3/4}}{\sqrt{1/u^5-11-u^5}}\\ =&5^{3/4}\sqrt{q}f^3(-q^5)/f^3(-q),\end{aligned}$$

by (2.13). Clearly, the last equality is equivalent to (4.10), and so the proof of (4.2) is complete. □

For the proof of (4.3), we need another transformation for incomplete elliptic integrals.

Lemma 4.4. *If* $0 < p < 1$ *and*

$$\tan\left(\frac{1}{2}(A-B)\right) = \frac{1-p}{1+2p}\tan B, \tag{4.13}$$

then

$$(1+2p)\int_0^A \frac{d\varphi}{\sqrt{1-p^3\left(\frac{2+p}{1+2p}\right)\sin^2\varphi}} = 3\int_0^B \frac{d\varphi}{\sqrt{1-p\left(\frac{2+p}{1+2p}\right)^3\sin^2\varphi}}.$$

This lemma is Entry 6(iv) in Chapter 19 in Ramanujan's second notebook and is a consequence of a Theorem of Jacobi; see [1, pp. 238–241] for a proof.

Proof of (4.3). We apply Lemma 4.4 with

$$p = \frac{1}{\epsilon^2\sqrt{5}},$$

where $\epsilon = (\sqrt{5}+1)/2$. Then

$$1+2p = \frac{3}{\sqrt{5}} \qquad \text{and} \qquad 2+p = \frac{3\epsilon}{\sqrt{5}}, \tag{4.14}$$

and so

$$p^3\left(\frac{2+p}{1+2p}\right) = \epsilon^{-5}5^{-3/2} \qquad \text{and} \qquad p\left(\frac{2+p}{1+2p}\right)^3 = \frac{\epsilon}{\sqrt{5}}.$$

If we substitute these quantities in Lemma 4.4, and if we set

$$A = 2\tan^{-1}\left(5^{3/4}\sqrt{q}f^3(-q^5)/f^3(-q)\right) \tag{4.15}$$

and

$$B = 2\tan^{-1}\left(5^{1/4}\sqrt{q}\psi(q^5)/\psi(q)\right), \tag{4.16}$$

we shall be finished with the proof of (4.3) if we can prove (4.13).

Using the subtraction formula for the tangent function, (4.15), and (4.16), we deduce that

$$\tan\left(\frac{1}{2}(A-B)\right) = \frac{5^{3/4}\sqrt{q}f^3(-q^5)/f^3(-q) - 5^{1/4}\sqrt{q}\psi(q^5)/\psi(q)}{1+5q\dfrac{f^3(-q^5)\psi(q^5)}{f^3(-q)\psi(q)}}. \tag{4.17}$$

It will be convenient to use some results from the lost notebook proved by Kang [10]. Set

$$t = q^{1/6}\frac{(-q^5;q^5)_\infty}{(-q;q)_\infty} \qquad \text{and} \qquad s = \frac{\varphi(-q)}{\varphi(-q^5)}, \tag{4.18}$$

where $\varphi(q)$ is defined by (2.2). Then

$$\frac{f(-q)}{q^{1/6}f(-q^5)}=\frac{s}{t} \qquad \text{and} \qquad \frac{\psi(q)}{\sqrt{q}\psi(q^5)}=\frac{s}{t^3}. \tag{4.19}$$

Employing (4.19) in (4.17), we readily deduce that

$$5^{-1/4}\tan\left(\frac{1}{2}(A-B)\right)=\frac{\sqrt{5}t^3s-t^3s^3}{s^4+5t^6}. \tag{4.20}$$

Next, a simple calculation shows that

$$1-p=\frac{3}{\epsilon\sqrt{5}}. \tag{4.21}$$

Hence, by (4.14), (4.21), (4.16), and the double angle formula,

$$\begin{aligned}5^{-1/4}\frac{1-p}{1+2p}\tan B =&5^{-1/4}\epsilon^{-1}\tan B\\ =&5^{-1/4}\epsilon^{-1}\tan\left(2\tan^{-1}\left(5^{1/4}\sqrt{q}\psi(q^5)/\psi(q)\right)\right)\\ =&\frac{2\epsilon^{-1}\sqrt{q}\psi(q^5)/\psi(q)}{1-\sqrt{5}\;q\psi^2(q^5)/\psi^2(q)}\\ =&\frac{2\epsilon^{-1}t^3s}{s^2-\sqrt{5}\;t^6}.\end{aligned} \tag{4.22}$$

Comparing (4.20) and (4.22), in view of (4.13), we must prove that

$$\frac{2\epsilon^{-1}t^3s}{s^2-\sqrt{5}\;t^6}=\frac{\sqrt{5}\;t^3s-t^3s^3}{s^4+5t^6}.$$

After considerable simplification, the last equality is seen to be equivalent to

$$s^4+5t^6=s^2+s^2t^6. \tag{4.23}$$

Now, from (4.18) and (2.3), we find that

$$t=t(q)=\frac{q^{1/6}\psi(q^5)f(-q^2)}{\psi(q)f(-q^{10})}.$$

Replacing q by $-q$ and employing (2.6) and (2.7), we find that

$$t^6(-q)=-q\left(\frac{\psi(-q^5)f(-q^2)}{\psi(-q)f(-q^{10})}\right)^6=-\left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/4},$$

where β has degree 5 over α. On the other hand, from (4.18),

$$s(-q)=\frac{\varphi(q)}{\varphi(q^5)}=:\sqrt{m},$$

where m is the multiplier of degree 5. Hence, replacing q by $-q$ in (4.23), we see that this equality is equivalent to

$$m^2-5\left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/4}=m-m\left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/4}. \tag{4.24}$$

Using formulas for m and $5/m$ given in Entry 13(xii) of Chapter 19 in Ramanujan's second notebook [1, pp. 281–282], namely,

$$m = \left(\frac{\beta}{\alpha}\right)^{1/4} + \left(\frac{1-\beta}{1-\alpha}\right)^{1/4} - \left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/4}$$

and

$$\frac{5}{m} = \left(\frac{\alpha}{\beta}\right)^{1/4} + \left(\frac{1-\alpha}{1-\beta}\right)^{1/4} - \left(\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right)^{1/4},$$

we may easily verify that (4.24) does hold to complete the proof. □

5. Elliptic Integrals of Order 5 (II)

Theorem 5.1 (p. 52)**.** *As before, let* $\epsilon = (\sqrt{5}+1)/2$, *and let* $u(q)$ *and* $f(-q)$ *be defined by (2.11) and (2.4), respectively. Then*

(5.1)
$$5^{-3/4}\int_0^q \frac{f^5(-t)}{\sqrt{f(-t^{1/5})f(-t^5)}}\frac{dt}{t^{9/10}} = 2\int_{\cos^{-1}\left(\sqrt{\epsilon u}\right)}^{\pi/2} \frac{d\varphi}{\sqrt{1-\epsilon^{-1}5^{-1/2}\sin^2\varphi}}$$

$$= \int_0^{2\tan^{-1}\left(5^{1/4}q^{1/10}\sqrt{f(-q^5)/f(-q^{1/5})}\right)} \frac{d\varphi}{\sqrt{1-\epsilon^{-1}5^{-1/2}\sin^2\varphi}} \tag{5.2}$$

(5.3)
$$= \frac{1}{\sqrt{5}}\int_0^{2\tan^{-1}\left(5^{3/4}q^{1/10}\left(\frac{f(-q^{1/5})+q^{1/5}f(-q^5)}{f(-q^{1/5})+5q^{1/5}f(-q^5)}\right)\sqrt{\frac{f(-q^5)}{f(-q^{1/5})}}\right)} \frac{d\varphi}{\sqrt{1-\epsilon^5 5^{-3/2}\sin^2\varphi}}.$$

Proof of (5.1). Let

$$\cos^2\varphi = \epsilon u(t). \tag{5.4}$$

Thus, if $t = 0$, then $\varphi = \pi/2$; if $t = q$, then $\varphi = \cos^{-1}\left(\sqrt{\epsilon u}\right)$. Upon differentiation and the use of Lemma 4.2,

$$2\cos\varphi(-\sin\varphi)\frac{d\varphi}{dt} = \epsilon u'(t) = \epsilon\frac{u(t)}{5t}\frac{f^5(-t)}{f(-t^5)}. \tag{5.5}$$

Therefore, by (5.5), (2.12), and (5.4),

$$\begin{aligned}
&5^{-3/4}\int_0^q \frac{f^5(-t)}{\sqrt{f(-t^{1/5})f(-t^5)}}\frac{dt}{t^{9/10}}\\
=&2\cdot 5^{1/4}\int_{\cos^{-1}\left(\sqrt{\epsilon u}\right)}^{\pi/2}\sqrt{\frac{t^{1/5}f(-t^5)}{f(-t^{1/5})}}\frac{\sin\varphi\ \cos\varphi}{\epsilon u(t)}d\varphi\\
=&2\cdot 5^{1/4}\int_{\cos^{-1}\left(\sqrt{\epsilon u}\right)}^{\pi/2}\frac{1}{\sqrt{1/u(t)-1-u(t)}}\frac{\sin\varphi}{\cos\varphi}d\varphi\\
=&2\cdot 5^{1/4}\int_{\cos^{-1}\left(\sqrt{\epsilon u}\right)}^{\pi/2}\frac{\sin\varphi}{\sqrt{\epsilon-\cos^2\varphi-\epsilon^{-1}\cos^4\varphi}}d\varphi. \qquad (5.6)
\end{aligned}$$

Now,

$$\begin{aligned}
\epsilon-\cos^2\varphi-\epsilon^{-1}\cos^4\varphi =&\epsilon-(1-\sin^2\varphi)-\epsilon^{-1}\cos^4\varphi\\
=&\epsilon^{-1}+\sin^2\varphi-\epsilon^{-1}\cos^4\varphi\\
=&\epsilon^{-1}(1-\cos^2\varphi)(1+\cos^2\varphi)+\sin^2\varphi\\
=&\epsilon^{-1}\sin^2\varphi(2-\sin^2\varphi)+\sin^2\varphi\\
=&\sin^2\varphi(2\epsilon^{-1}-\epsilon^{-1}\sin^2\varphi+1)\\
=&\sin^2\varphi(\sqrt{5}-\epsilon^{-1}\sin^2\varphi).
\end{aligned}$$

Using this calculation in (5.6), we find that

$$5^{-3/4}\int_0^q \frac{f^5(-t)}{\sqrt{f(-t^{1/5})f(-t^5)}}\frac{dt}{t^{9/10}} = 2\cdot 5^{1/4}\int_{\cos^{-1}\left(\sqrt{\epsilon u}\right)}^{\pi/2}\frac{d\varphi}{\sqrt{\sqrt{5}-\epsilon^{-1}\sin^2\varphi}},$$

from which (5.1) is immediate. □

Proof of (5.2). The proof is similar to that of (4.2). We begin with (4.9), set $x=\epsilon^{-1}5^{-1/2}$, and put $\gamma=\frac{\pi}{2}-\cos^{-1}\left(\sqrt{\epsilon u}\right)$. Thus,

$$\tan\gamma = \cot\left(\cos^{-1}\left(\sqrt{\epsilon u}\right)\right) = \sqrt{\frac{\epsilon u}{1-\epsilon u}}. \qquad (5.7)$$

As with the proof of (4.2), we want to show that conditions (i) and (ii) imply that

$$\beta = 2\tan^{-1}\left(5^{1/4}q^{1/10}\sqrt{f(-q^5)/f(-q^{1/5})}\right). \qquad (5.8)$$

From condition (ii) and (5.7),

$$\tan\alpha = \frac{\tan\gamma}{\sqrt{1-\epsilon^{-1}5^{-1/2}}} = \sqrt{\frac{\sqrt{5}\ u}{1-\epsilon u}} \qquad (5.9)$$

and

$$\sin^2\alpha = \frac{\tan^2\alpha}{1+\tan^2\alpha} = \frac{\sqrt{5}\ u}{1+\epsilon^{-1}u}. \qquad (5.10)$$