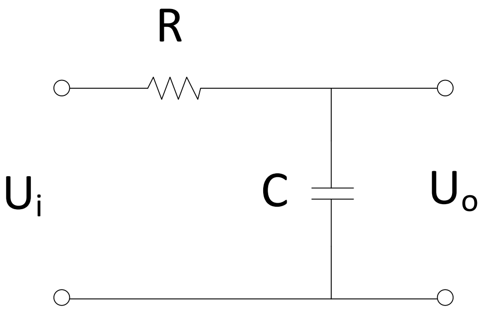
Control Laboratory

Chapter 8 Frequency Response Methods

8.1 Introduction

Example:

If $U_i = A\sin\omega t$, determine $y(t) = U_o(t)$



$$\frac{Y(s)}{R(s)} = \frac{U_o(s)}{U_i(s)} = \frac{1}{RCs+1}$$

$$Y(s) = \frac{1}{RCs+1}R(s) = \frac{1}{RCs+1}\cdot\frac{A\omega}{s^2+\omega^2}$$

$$y(t) = \frac{A\tau\omega}{1+\tau^2\omega^2}e^{-\frac{t}{\tau}} + \frac{A}{\sqrt{1+\tau^2\omega^2}}\sin(\omega t - \arctan\tau\omega) \quad (\tau = RC)$$

$$y(\infty) = \lim_{t\to\infty} y(t) = \frac{A}{\sqrt{1+\tau^2\omega^2}}\sin(\omega t - \arctan\tau\omega)$$

8.1 Introduction

$$\frac{A(\omega)}{A} = \frac{1}{\sqrt{1+\tau^2\omega^2}}, \qquad \Phi(\omega) = -\arctan\tau\omega$$

amplitude **phase**

Both are the function of ω

$G(s) = \dfrac{1}{\tau s+1}$ Let $s = j\omega$, we get frequency transfer function

$$G(j\omega) = \frac{1}{j\tau\omega+1}$$

$$|G(j\omega)| = \frac{1}{\sqrt{\tau^2\omega^2+1}}$$

amplitude character

$$\angle G(j\omega) = -\arctan\omega\tau$$

phase character

8.1 Introduction

Concepts:

The frequency response of a system is defined as the steady-state response of the system to a sinusoidal input signal

Note:

① the sinusoidal is a unique input signal;

② For a linear system, output signal is sinusoidal in the steady state. (Frequency method is for steady, not dynamic process)

③ Compared with the input sinusoidal signal $A\sin(\omega t)$, the magnitude increases $|T(j\omega)|$, and the phase changes $\angle T(j\omega)$

Advantages:

① Frequency characteristic plot can be measured by experiment;

② Plot is convenient for analysis and design.

8.2 Frequency Response Plots

System output response

$$Y(jw) = T(jw)R(jw) = \frac{G(jw)}{1+G(jw)}R(jw)$$

System close-loop transfer function

$$T(jw) = \frac{G(jw)}{1+G(jw)}$$

System open-loop transfer function

$$T(jw) = \frac{G(jw)}{1+G(jw)} \Rightarrow G(jw)$$

8.2 Frequency Response Plots

System open-loop transfer function

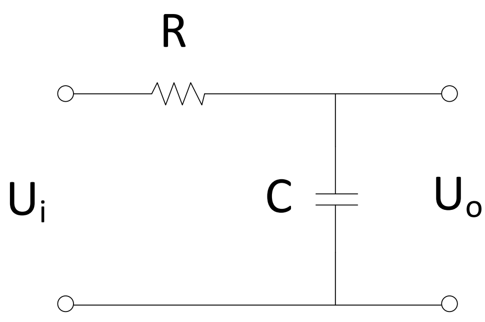
$$G(s)\overset{s=j\omega}{\Longrightarrow}G(j\omega)=R(\omega)+jX(\omega)=|G(j\omega)|e^{\angle G(j\omega)}$$

$$R(\omega)=\mathrm{Re}[G(j\omega)],\ jX(\omega)=\mathrm{Im}[G(j\omega)]$$

- **polar plot**

$$G(s)=\frac{1}{RCs+1}\Rightarrow G(j\omega)=\frac{1}{RC\omega j+1}$$

$$\text{let}\,\omega_1=\frac{1}{RC}\Rightarrow G(j\omega)=\frac{1}{\frac{\omega}{\omega_1}j+1}$$



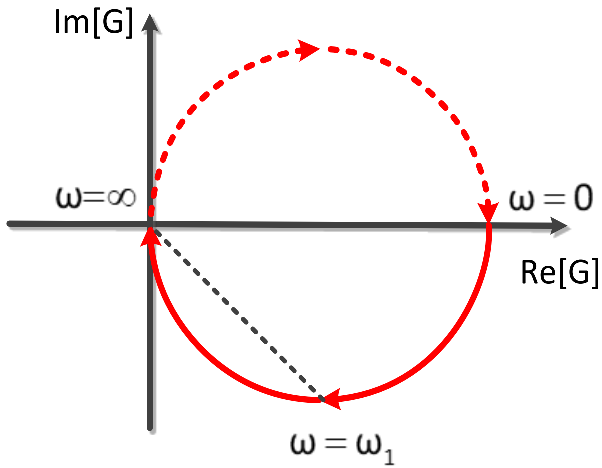
8.2 Frequency Response Plots

$$G(j\omega)=\frac{1}{1+(\frac{\omega}{\omega_1})^2}-j\frac{\frac{\omega}{\omega_1}}{1+(\frac{\omega}{\omega_1})^2}$$

$$|G(j\omega)|=\frac{1}{\sqrt{1+(\frac{\omega}{\omega_1})^2}}$$

$$\angle G(j\omega)=-\arctan\frac{\omega}{\omega_1}$$

ω	$R(\omega)$	$X(\omega)$	$\lvert G(j\omega)\rvert$	$\angle G(j\omega)$
0	1	0	1	0
ω_1	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$-45°$
∞	0	0	0	$-90°$



8.2 Frequency Response Plots

Tips:

① $R(\omega), X(\omega)$ or $|G(j\omega)|, \angle G(j\omega)$, choose one method for drawing

② $\omega(-\infty \to +\infty)$ so the plot is symmetric about the real axis;

the regular polar plot is from 0 to +∞, we use the plot of $\omega(-\infty \to +\infty)$ to verify the stability of system.

Example:

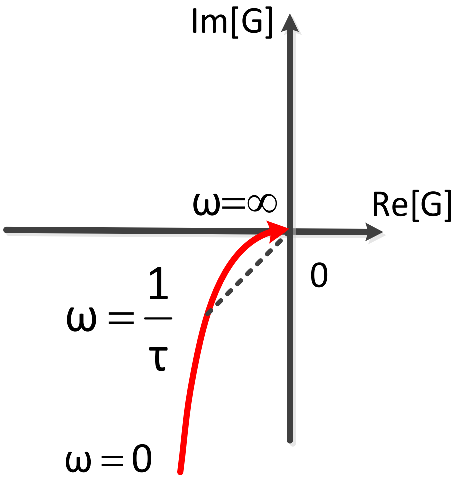
$$G(s) = \frac{1}{s(s+1)} \Rightarrow G(j\omega) = \frac{1}{jw(jw\tau+1)}$$

$$|G(j\omega)| = \frac{1}{|jw||jw\tau+1|} = \frac{1}{w\sqrt{(w\tau)^2+1}};$$

$$\angle G(j\omega) = -90° - \tan^{-1}\tau\omega$$

8.2 Frequency Response Plots

ω	$\lvert G(j\omega)\rvert$	$\angle G(j\omega)$
0	∞	-90°
$\frac{1}{\tau}$	$\frac{1}{\sqrt{2}}$	-135°
∞	0	-180°



The limitations of polar plot:

① The addition of poles and zeros to an existing system requires the recalculation of the frequency response.

② Calculating the frequency response in this manner is tedious and does not indicate the effect of the individual poles and zeros.

8.2 Frequency Response Plots

Unit: decibels(dB)

Ⅱ logarithmic plots (Bode diagram)

Draw the diagram of $G(j\omega) \Rightarrow 20\lg|G(j\omega)|$ and $\angle G(j\omega)$

Eg: $G(j\omega) = \dfrac{1}{j\omega\tau + 1}$

$$20\lg|G(j\omega)| = 20\lg\frac{1}{\sqrt{\tau^2\omega^2+1}} = -10\lg(\tau^2\omega^2+1)$$

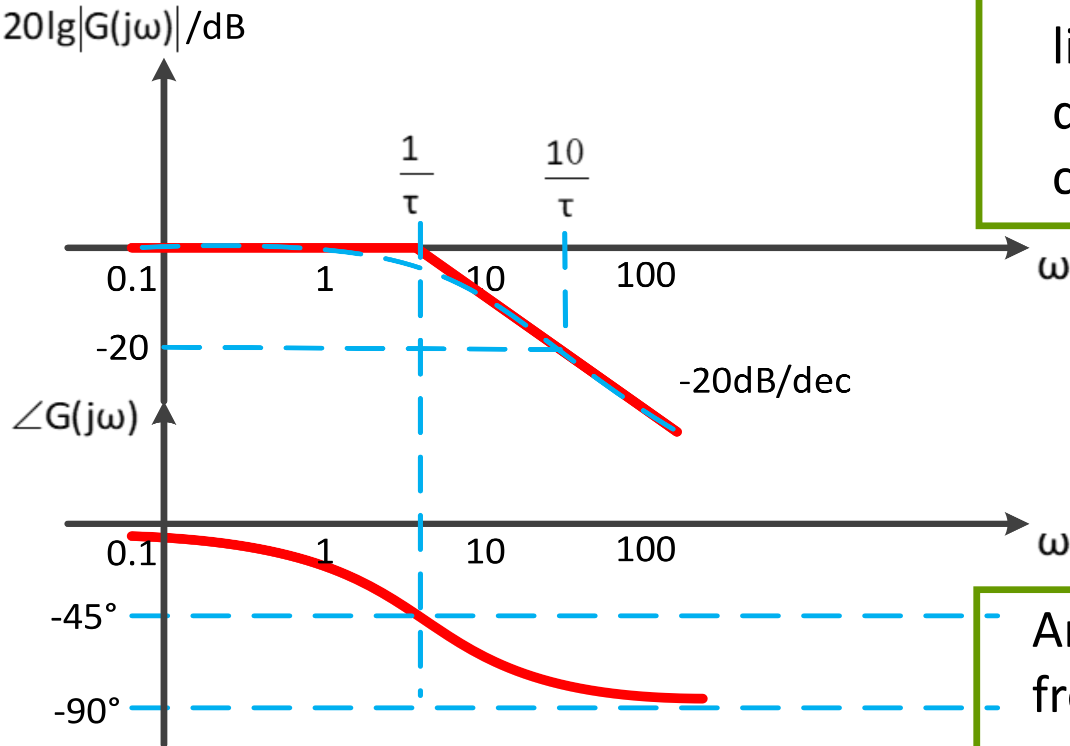
$$\omega << \frac{1}{\tau} \qquad 20\lg|G(j\omega)| = 0$$

$$\omega >> \frac{1}{\tau} \qquad 20\lg|G(j\omega)| = -20\lg\omega\tau$$

$$\omega = \frac{1}{\tau} \qquad 20\lg|G(j\omega)| = -20\lg\frac{1}{\sqrt{2}} = -3\text{dB}$$

Break frequency or corner frequency

8.2 Frequency Response Plots



Semilog paper with a linear coordinate for dB and a logarithmic coordinate for ω .

An interval of two frequencies with a ratio equal to 10 is called a decade.

8.2 Frequency Response Plots

Advantages of the logarithmic coordinate:

① The extension of the range of frequencies;

② Multiplicative factors are converted into additive factors.

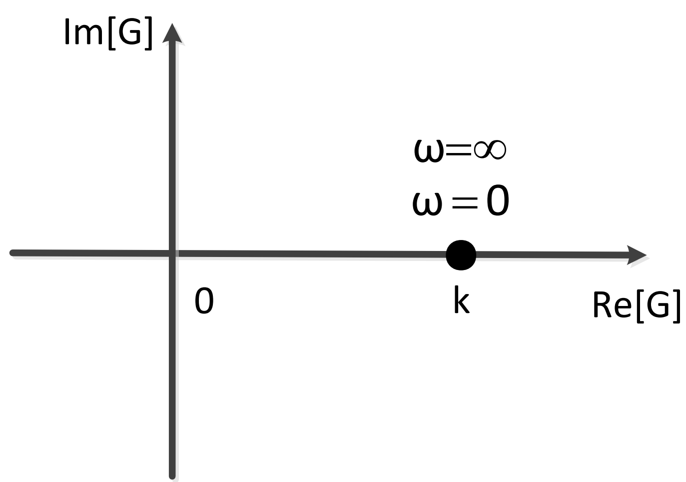
Note: $\lg 0 = -\infty$, so there is no $\omega = 0$ point on the horizontal axis

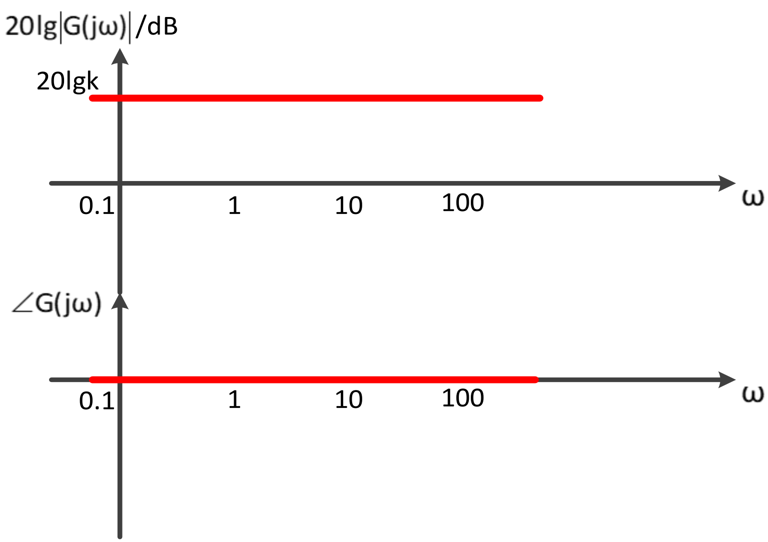
8.2 Frequency Response Plots

Typical elements :

①**proportional element**

$$G(s)=k, G(j\omega)=k, |G(j\omega)|=k, \angle G(j\omega)=0$$

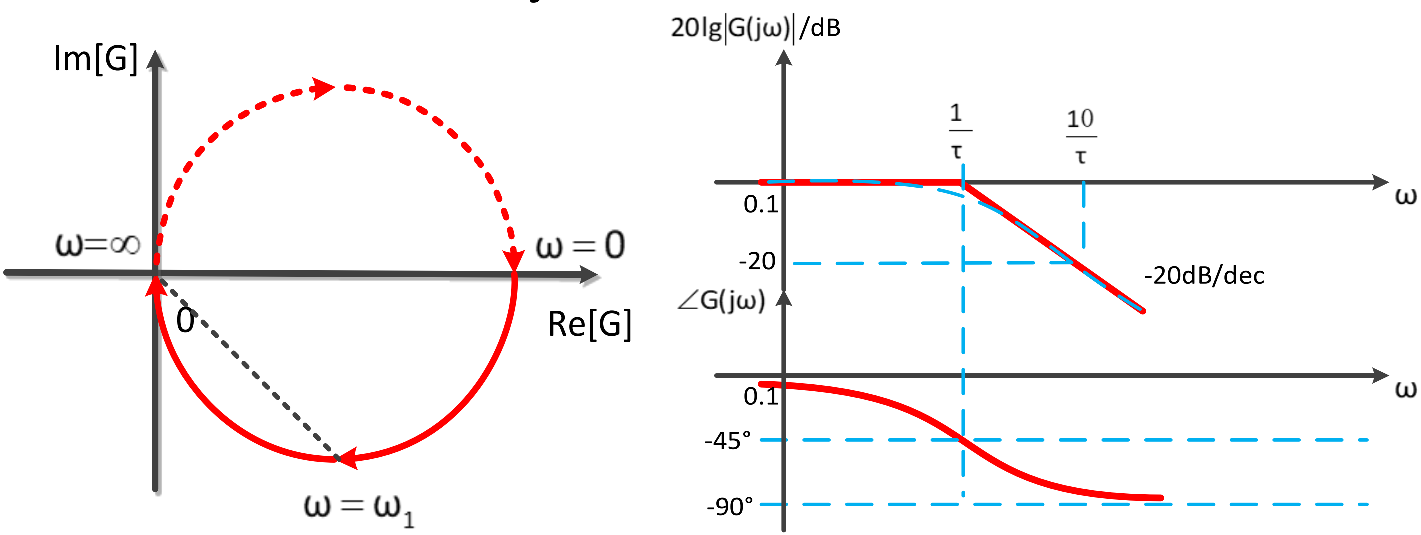




8.2 Frequency Response Plots

②inertial element

$$G(s)=\frac{1}{\tau s+1},G(j\omega)=\frac{1}{j\omega\tau+1}=\frac{1}{\omega^2\tau^2+1}-j\frac{\omega\tau}{\omega^2\tau^2+1}$$



The magnitude curve is asymptotic, the biggest error is 3dB at $\frac{1}{\tau}$

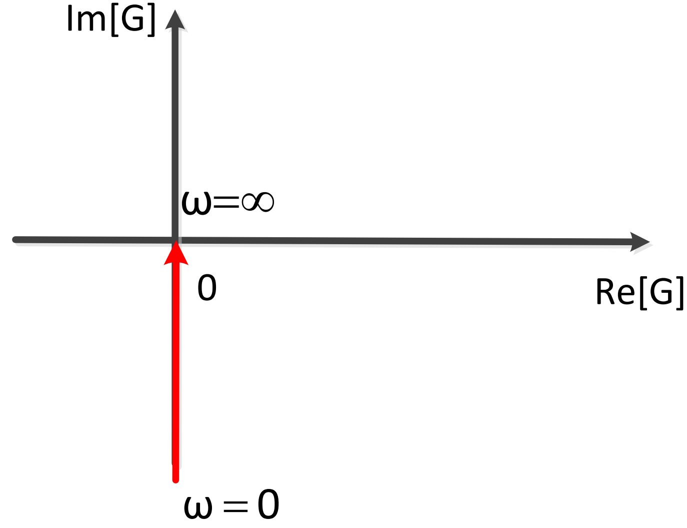
$\omega=\frac{1}{\tau}$:break frequency or corner frequency

8.2 Frequency Response Plots

③**integral element**

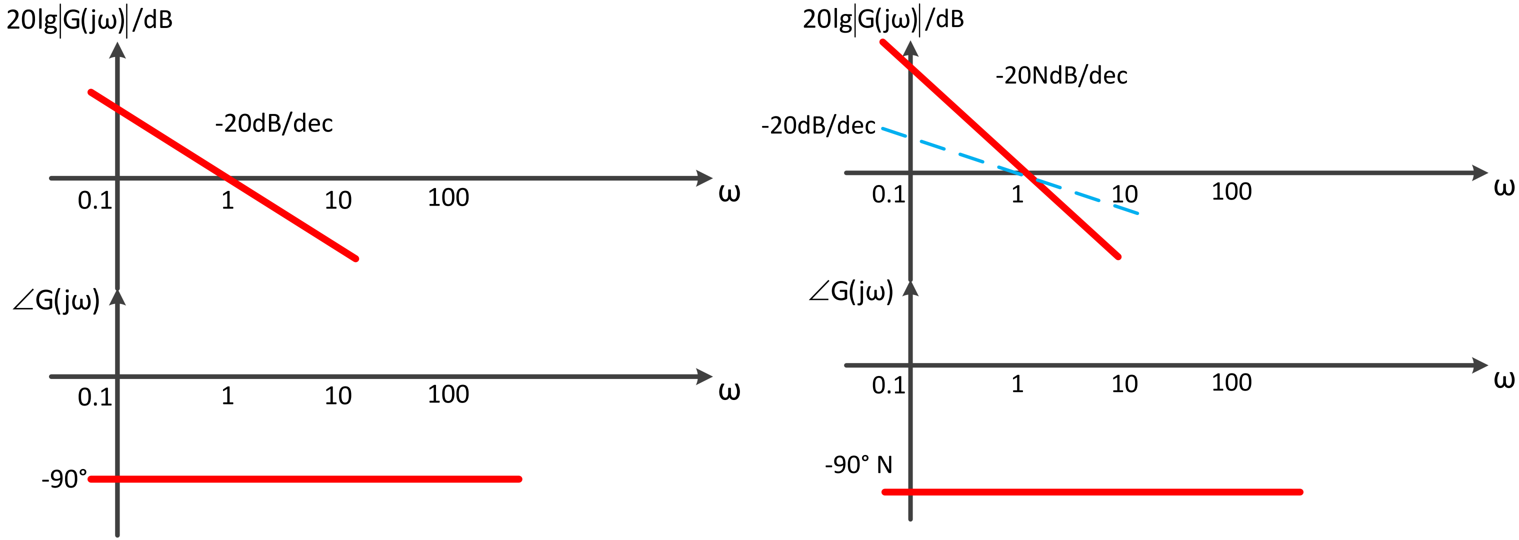
$$G(s)=\frac{1}{s},G(j\omega)=\frac{1}{j\omega}=-j\frac{1}{\omega},\left|G(j\omega)\right|=\frac{1}{\omega},\angle G(j\omega)=-90^\circ$$

ω	$\left\|G(j\omega)\right\|$	$\angle G(j\omega)$
0	∞	-90°
∞	0	-90°



$$20\lg\left|G(j\omega)\right|=20\lg\frac{1}{\omega}=-20\lg\omega$$

8.2 Frequency Response Plots



$$G(s)=\frac{1}{s^N},G(j\omega)=\frac{1}{(j\omega)^N},\angle G(j\omega)=\angle j\omega+\angle j\omega+\cdots=-90^\circ\times N$$

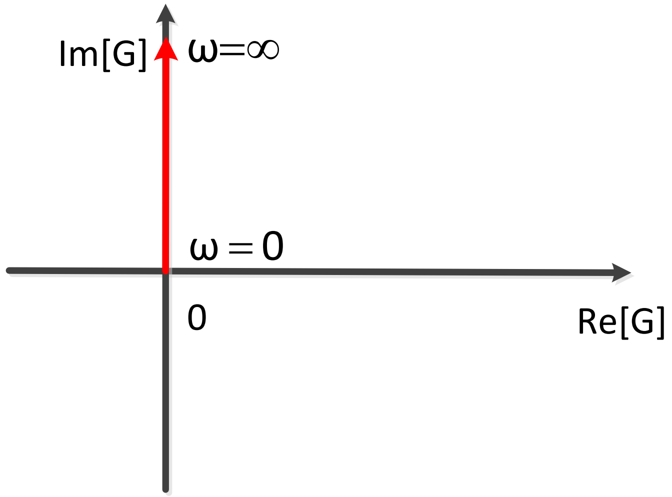
$$|G(j\omega)|=\frac{1}{\omega^N},20\lg|G(j\omega)|=-20N\lg\omega$$

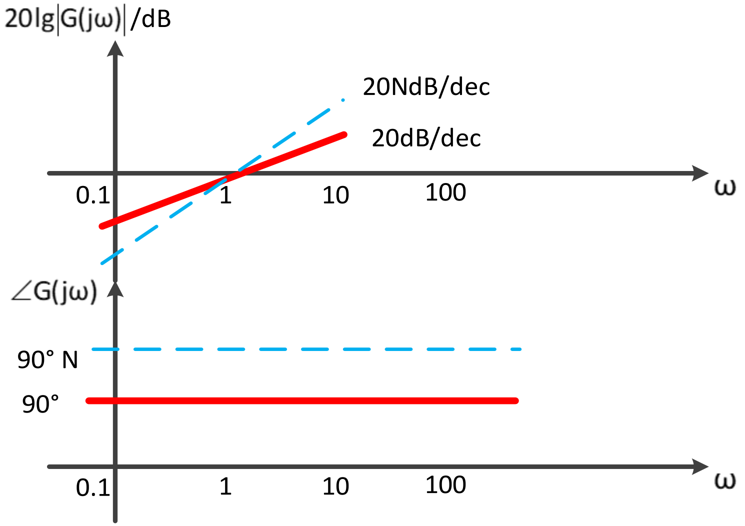
8.2 Frequency Response Plots

④**differential element**

$$G(s)=s, G(j\omega)=j\omega, |G(j\omega)|=\omega, \angle G(j\omega)=90^\circ$$

$$20\lg|G(j\omega)|=20\lg\omega$$





8.2 Frequency Response Plots

⑤**first-order differential element**

$$G(s)=\tau s+1, G(j\omega)=j\omega\tau+1 \overset{\omega_1=\frac{1}{\tau}}{\Rightarrow} G(j\omega)=j\frac{\omega}{\omega_1}+1, |G(j\omega)|=\sqrt{1+(\frac{\omega}{\omega_1})^2}$$

$$\angle G(j\omega)=\arctan\frac{\omega}{\omega_1},\ 20\lg|G(j\omega)|=20\lg\sqrt{1+(\frac{\omega}{\omega_1})^2}$$

ω	$20\lg\sqrt{1+(\frac{\omega}{\omega_1})^2}$
$\omega \ll \omega_1=\frac{1}{\tau}$	0
$\omega \gg \omega_1=\frac{1}{\tau}$	∞

ω	$\lvert G(j\omega)\rvert$	$\angle G(j\omega)$
0	1	0
$\frac{1}{\tau}$	$\sqrt{2}$	45°
∞	∞	90°

8.2 Frequency Response Plots