40

QUANTUM MECHANICS

40.1. **IDENTIFY:** Using the momentum of the free electron, we can calculate k and ω and use these to express its wave function.

SET UP: $\Psi(x,t) = Ae^{ikx}e^{-i\omega t}$, $k = p/\hbar$, and $\omega = \hbar k^2/2m$.

EXECUTE: $k = \frac{p}{\hbar} = -\frac{4.50\times 10^{-24}\ \text{kg}\cdot\text{m/s}}{1.055\times 10^{-34}\ \text{J}\cdot\text{s}} = -4.27\times 10^{10}\ \text{m}^{-1}$.

$$\omega = \frac{\hbar k^2}{2m} = \frac{(1.055\times 10^{-34}\ \text{J}\cdot\text{s})(4.27\times 10^{10}\ \text{m}^{-1})^2}{2(9.108\times 10^{-31}\ \text{kg})} = 1.05\times 10^{17}\ \text{s}^{-1}.$$

$$\Psi(x,t) = Ae^{-i[4.27\times 10^{10}\ \text{m}^{-1})x}e^{-i[1.05\times 10^{17}\ \text{s}^{-1}]t}.$$

EVALUATE: The wave function depends on position and time.

40.2. **IDENTIFY:** Using the known wave function for the particle, we want to find where its probability function is a maximum.

SET UP: $|\Psi(x,t)|^2 = |A|^2[e^{ikx}e^{-i\omega t} - e^{2ikx}e^{-4i\omega t}][e^{-ikx}e^{+i\omega t} - e^{-2ikx}e^{+4i\omega t}]$.

$|\Psi(x,t)|^2 = |A|^2(2-[e^{-i(kx-3\omega t)} + e^{+i(kx-3\omega t)}]) = 2|A|^2(1-\cos(kx-3\omega t))$.

EXECUTE: **(a)** For $t=0$, $|\Psi(x,t)|^2 = 2|A|^2(1-\cos(kx))$. $|\Psi(x,t)|^2$ is a maximum when $\cos(kx) = -1$ and this happens when $kx = (2n+1)\pi$, $n = 0,1,\ldots$. $|\Psi(x,t)|^2$ is a maximum for $x = \frac{\pi}{k}, \frac{3\pi}{k}$, etc.

(b) $t = \frac{2\pi}{\omega}$ and $3\omega t = 6\pi$. $|\Psi(x,t)|^2 = 2|A|^2(1-\cos(kx-6\pi))$. Maximum for $kx - 6\pi = \pi, 3\pi, \ldots$, which gives maxima when $x = \frac{7\pi}{k}, \frac{9\pi}{k}$.

(c) From the results for parts (a) and (b), $v_{\text{av}} = \frac{7\pi/k - \pi/k}{2\pi/\omega} = \frac{3\omega}{k}$. $v_{\text{av}} = \frac{\omega_2 - \omega_1}{k_2 - k_1}$ with $\omega_2 = 4\omega$, $\omega_1 = \omega$, $k_2 = 2k$ and $k_1 = k$ gives $v_{\text{av}} = \frac{3\omega}{k}$.

EVALUATE: The expressions in part (c) agree.

40.3. **IDENTIFY:** Use the wave function from Example 40.1.

SET UP: $|\Psi(x,t)|^2 = 2|A|^2\{1+\cos[(k_2-k_1)x - (\omega_2-\omega_1)t]\}$. $k_2 = 3k_1 = 3k$. $\omega = \frac{\hbar k^2}{2m}$, so $\omega_2 = 9\omega_1 = 9\omega$.

$|\Psi(x,t)|^2 = 2|A|^2\{1+\cos(2kx - 8\omega t)\}$.

EXECUTE: **(a)** At $t = 2\pi/\omega$, $|\Psi(x,t)|^2 = 2|A|^2\{1+\cos(2kx-16\pi)\}$. $|\Psi(x,t)|^2$ is maximum for $\cos(2kx-16\pi) = 1$. This happens for $2kx - 16\pi = 0, 2\pi, \ldots$. Smallest positive x where $|\Psi(x,t)|^2$ is a maximum is $x = \frac{8\pi}{k}$.

(b) From the result of part (a), $v_{av} = \dfrac{8\pi/k}{2\pi/\omega} = \dfrac{4\omega}{k}$. $v_{av} = \dfrac{\omega_2 - \omega_1}{k_2 - k_1} = \dfrac{8\omega}{2k} = \dfrac{4\omega}{k}$.

EVALUATE: The two expressions agree.

40.4. **IDENTIFY:** We have a free particle, described in Example 40.1.

SET UP and EXECUTE: $v_{av} = \dfrac{\omega_2 - \omega_1}{k_2 - k_1} = \dfrac{\hbar}{2m}\dfrac{(k_2^2 - k_1^2)}{k_2 - k_1} = \dfrac{\hbar}{2m}\dfrac{(k_2 + k_1)(k_2 - k_1)}{k_2 - k_1} = \dfrac{\hbar}{2m}(k_2 + k_1) = \dfrac{p_{av}}{m}$.

EVALUATE: This is the same as the classical physics result, $v = p/m = mv/m = v$.

40.5. **IDENTIFY and SET UP:** $\psi(x) = A\sin kx$. The position probability density is given by $|\psi(x)|^2 = A^2\sin^2 kx$.

EXECUTE: **(a)** The probability is highest where $\sin kx = 1$ so $kx = 2\pi x/\lambda = n\pi/2,\ n = 1, 3, 5, \ldots$

$x = n\lambda/4,\ n = 1, 3, 5, \ldots$ so $x = \lambda/4, 3\lambda/4, 5\lambda/4, \ldots$

(b) The probability of finding the particle is zero where $|\psi|^2 = 0$, which occurs where $\sin kx = 0$ and

$kx = 2\pi x/\lambda = n\pi,\ n = 0, 1, 2, \ldots$

$x = n\lambda/2,\ n = 0, 1, 2, \ldots$ so $x = 0, \lambda/2, \lambda, 3\lambda/2, \ldots$

EVALUATE: The situation is analogous to a standing wave, with the probability analogous to the square of the amplitude of the standing wave.

40.6. **IDENTIFY and SET UP:** $|\Psi|^2 = \Psi^*\Psi$

EXECUTE: $\Psi^* = \psi^*\sin\omega t$, so $|\Psi|^2 = \Psi^*\Psi = \psi^*\psi\sin^2\omega t = |\psi|^2\sin^2\omega t$. $|\Psi|^2$ is not time-independent, so Ψ is not the wavefunction for a stationary state.

EVALUATE: $\Psi = \psi e^{i\omega\phi} = \psi(\cos\omega t + i\sin\omega t)$ is a wavefunction for a stationary state, since for it $|\Psi|^2 = |\psi|^2$, which is time independent.

40.7. **IDENTIFY:** Determine whether or not $-\dfrac{\hbar^2}{2m}\dfrac{d^2\psi}{dx^2} + U\psi$ is equal to $E\psi$, for some value of E.

SET UP: $-\dfrac{\hbar^2}{2m}\dfrac{d^2\psi_1}{dx^2} + U\psi_1 = E_1\psi_1$ and $-\dfrac{\hbar^2}{2m}\dfrac{d^2\psi_2}{dx^2} + U\psi_2 = E_2\psi_2$

EXECUTE: $-\dfrac{\hbar^2}{2m}\dfrac{d^2\psi}{dx^2} + U\psi = BE_1\psi_1 + CE_2\psi_2$. If ψ were a solution with energy E, then $BE_1\psi_1 + CE_2\psi_2 = BE\psi_1 + CE\psi_2$ or $B(E_1 - E)\psi_1 = C(E - E_2)\psi_2$. This would mean that ψ_1 is a constant multiple of ψ_2, and ψ_1 and ψ_2 would be wave functions with the same energy. However, $E_1 \neq E_2$, so this is not possible, and ψ cannot be a solution to Eq. (40.23).

EVALUATE: ψ is a solution if $E_1 = E_2$; see Exercise 40.9.

40.8. **IDENTIFY:** Apply the Heisenberg Uncertainty Principle in the form $\Delta x\Delta p_x \geq \hbar/2$.

SET UP: The uncertainty in the particle position is proportional to the width of $\psi(x)$.

EXECUTE: The width of $\psi(x)$ is inversely proportional to $\sqrt{\alpha}$. This can be seen by either plotting the function for different values of α or by finding the full width at half-maximum. The particle's uncertainty in position decreases with increasing α.

(b) Since the uncertainty in position decreases, the uncertainty in momentum must increase.

EVALUATE: As α increases, the function $A(k)$ in Eq. (40.19) must become broader.

40.9. **IDENTIFY:** Determine whether or not $-\dfrac{\hbar^2}{2m}\dfrac{d^2\psi}{dx^2} + U\psi$ is equal to $E\psi$.

SET UP: ψ_1 and ψ_2 are solutions with energy E means that $-\dfrac{\hbar^2}{2m}\dfrac{d^2\psi_1}{dx^2} + U\psi_1 = E\psi_1$ and

$-\dfrac{\hbar^2}{2m}\dfrac{d^2\psi_2}{dx^2} + U\psi_2 = E\psi_2$.

EXECUTE: Eq. (40.23): $\frac{-\hbar^2}{2m}\frac{d^2\psi}{dx^2}+U\psi=E\psi$. Let $\psi=A\psi_1+B\psi_2$

$$\Rightarrow \frac{-\hbar^2}{2m}\frac{d^2}{dx^2}(A\psi_1+B\psi_2)+U(A\psi_1+B\psi_2)=E(A\psi_1+B\psi_2)$$

$\Rightarrow A\left(-\frac{\hbar^2}{2m}\frac{d^2\psi_1}{dx^2}+U\psi_1-E\psi_1\right)+B\left(-\frac{\hbar^2}{2m}\frac{d^2\psi_2}{dx^2}+U\psi_2-E\psi_2\right)=0.$ But each of ψ_1 and ψ_2 satisfy Schrödinger's equation separately so the equation still holds true, for any A or B.

EVALUATE: If ψ_1 and ψ_2 are solutions of the Schrodinger equation for different energies, then $\psi=B\psi_1+C\psi_2$ is not a solution (Exercise 40.7).

40.10. **IDENTIFY:** To describe a real situation, a wave function must be normalizable.

SET UP: $|\psi|^2\,dV$ is the probability that the particle is found in volume dV. Since the particle must be *somewhere*, ψ must have the property that $\int|\psi|^2dV=1$ when the integral is taken over all space.

EXECUTE: **(a)** For normalization of the one-dimensional wave function, we have

$$1=\int_{-\infty}^{\infty}|\psi|^2dx=\int_{-\infty}^{0}(Ae^{bx})^2dx+\int_0^{\infty}(Ae^{-bx})^2dx=\int_{-\infty}^{0}A^2e^{2bx}dx+\int_0^{\infty}A^2e^{-2bx}dx.$$

$1=A^2\left\{\frac{e^{2bx}}{2b}\bigg|_{-\infty}^{0}+\frac{e^{-2bx}}{-2b}\bigg|_{0}^{\infty}\right\}=\frac{A^2}{b}$, which gives $A=\sqrt{b}=\sqrt{2.00\text{ m}^{-1}}=1.41\text{ m}^{-1/2}$

(b) The graph of the wavefunction versus x is given in Figure 40.10.

(c) (i) $P=\int_{-0.500\text{ m}}^{+5.00\text{ m}}|\psi|^2dx=2\int_0^{+5.00\text{ m}}A^2e^{-2bx}dx$, where we have used the fact that the wave function is an even function of x. Evaluating the integral gives

$$P=\frac{-A^2}{b}(e^{-2b(0.500\text{ m})}-1)=\frac{-(2.00\text{ m}^{-1})}{2.00\text{ m}^{-1}}(e^{-2.00}-1)=0.865$$

There is a little more than an 86% probability that the particle will be found within 50 cm of the origin.

(ii) $P=\int_{-\infty}^{0}(Ae^{bx})^2dx=\int_{-\infty}^{0}A^2e^{2bx}dx=\frac{A^2}{2b}=\frac{2.00\text{ m}^{-1}}{2(2.00\text{ m}^{-1})}=\frac{1}{2}=0.500$

There is a 50-50 chance that the particle will be found to the left of the origin, which agrees with the fact that the wave function is symmetric about the y-axis.

(iii) $P=\int_{0.500\text{ m}}^{1.00\text{ m}}A^2e^{-2bx}dx$

$$=\frac{A^2}{-2b}(e^{-2(2.00\text{ m}^{-1})(1.00\text{ m})}-e^{-2(2.00\text{ m}^{-1})(0.500\text{ m})})=-\frac{1}{2}(e^{-4}-e^{-2})=0.0585$$

EVALUATE: There is little chance of finding the particle in regions where the wave function is small.

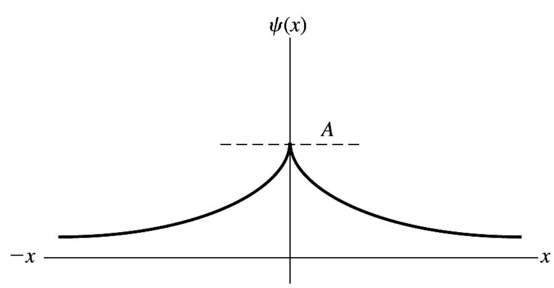

Figure 40.10

40.11. **IDENTIFY** and **SET UP:** The energy levels for a particle in a box are given by $E_n = \dfrac{n^2h^2}{8mL^2}$.

EXECUTE: **(a)** The lowest level is for $n = 1$, and $E_1 = \dfrac{(1)(6.626 \times 10^{-34}\ \text{J}\cdot\text{s})^2}{8(0.20\ \text{kg})(1.3\ \text{m})^2} = 1.6 \times 10^{-67}\ \text{J}$.

(b) $E = \dfrac{1}{2}mv^2$ so $v = \sqrt{\dfrac{2E}{m}} = \sqrt{\dfrac{2(1.2 \times 10^{-67}\ \text{J})}{0.20\ \text{kg}}} = 1.3 \times 10^{-33}\ \text{m/s}$. If the ball has this speed the time it would take it to travel from one side of the table to the other is

$t = \dfrac{1.3\ \text{m}}{1.3 \times 10^{-33}\ \text{m/s}} = 1.0 \times 10^{33}\ \text{s}$.

(c) $E_1 = \dfrac{h^2}{8mL^2}$, $E_2 = 4E_1$, so $\Delta E = E_2 - E_1 = 3E_1 = 3(1.6 \times 10^{-67}\ \text{J}) = 4.9 \times 10^{-67}\ \text{J}$.

(d) **EVALUATE:** No, quantum mechanical effects are not important for the game of billiards. The discrete, quantized nature of the energy levels is completely unobservable.

40.12. **IDENTIFY:** Solve Eq. (40.31) for L.

SET UP: The ground state has $n = 1$.

EXECUTE: $L = \dfrac{h}{\sqrt{8mE_1}} = \dfrac{(6.626 \times 10^{-34}\ \text{J}\cdot\text{s})}{\sqrt{8(1.673 \times 10^{-27}\ \text{kg})(5.0 \times 10^6\ \text{eV})(1.602 \times 10^{-19}\ \text{J/eV})}} = 6.4 \times 10^{-15}\ \text{m}$

EVALUATE: The value of L we calculated is on the order of the diameter of a nucleus.

40.13. **IDENTIFY:** An electron in the lowest energy state in this box must have the same energy as it would in the ground state of hydrogen.

SET UP: The energy of the n^{th} level of an electron in a box is $E_n = \dfrac{nh^2}{8mL^2}$.

EXECUTE: An electron in the ground state of hydrogen has an energy of $-13.6\ \text{eV}$, so find the width corresponding to an energy of $E_1 = 13.6\ \text{eV}$. Solving for L gives

$L = \dfrac{h}{\sqrt{8mE_1}} = \dfrac{(6.626 \times 10^{-34}\ \text{J}\cdot\text{s})}{\sqrt{8(9.11 \times 10^{-31}\ \text{kg})(13.6\ \text{eV})(1.602 \times 10^{-19}\ \text{J/eV})}} = 1.66 \times 10^{-10}\ \text{m}$.

EVALUATE: This width is of the same order of magnitude as the diameter of a Bohr atom with the electron in the K shell.

40.14. **IDENTIFY** and **SET UP:** The energy of a photon is $E = hf = h\dfrac{c}{\lambda}$. The energy levels of a particle in a box are given by Eq. (40.31).

EXECUTE: **(a)** $E = (6.63 \times 10^{-34}\ \text{J}\cdot\text{s})\dfrac{(3.00 \times 10^8\ \text{m/s})}{(122 \times 10^{-9}\ \text{m})} = 1.63 \times 10^{-18}\ \text{J}$. $\Delta E = \dfrac{h^2}{8mL^2}(n_1^2 - n_2^2)$.

$L = \sqrt{\dfrac{h^2(n_1^2 - n_2^2)}{8m\Delta E}} = \sqrt{\dfrac{(6.63 \times 10^{-34}\ \text{J}\cdot\text{s})^2(2^2 - 1^2)}{8(9.11 \times 10^{-31}\ \text{kg})(1.63 \times 10^{-18}\ \text{J})}} = 3.33 \times 10^{-10}\ \text{m}$.

(b) The ground state energy for an electron in a box of the calculated dimensions is

$E = \dfrac{h^2}{8mL^2} = \dfrac{(6.63 \times 10^{-34}\ \text{J}\cdot\text{s})^2}{8(9.11 \times 10^{-31}\ \text{kg})(3.33 \times 10^{-10}\ \text{m})^2} = 5.43 \times 10^{-19}\ \text{J} = 3.40\ \text{eV}$ (one-third of the original photon energy), which does not correspond to the $-13.6\ \text{eV}$ ground state energy of the hydrogen atom.

EVALUATE: **(c)** Note that the energy levels for a particle in a box are proportional to n^2, whereas the energy levels for the hydrogen atom are proportional to $-\dfrac{1}{n^2}$. A one-dimensional box is not a good model for a hydrogen atom.

40.15. **IDENTIFY** and **SET UP:** Eq. (40.31) gives the energy levels. Use this to obtain an expression for $E_2 - E_1$ and use the value given for this energy difference to solve for L.

EXECUTE: Ground state energy is $E_1 = \frac{h^2}{8mL^2}$; first excited state energy is $E_2 = \frac{4h^2}{8mL^2}$. The energy separation between these two levels is $\Delta E = E_2 - E_1 = \frac{3h^2}{8mL^2}$. This gives $L = h\sqrt{\frac{3}{8m\Delta E}} =$

$$L = 6.626\times10^{-34}\ \text{J}\cdot\text{s}\sqrt{\frac{3}{8(9.109\times10^{-31}\ \text{kg})(3.0\ \text{eV})(1.602\times10^{-19}\ \text{J/1 eV})}} = 6.1\times10^{-10}\ \text{m} = 0.61\ \text{nm}.$$

EVALUATE: This energy difference is typical for an atom and L is comparable to the size of an atom.

40.16. **IDENTIFY:** The energy of the absorbed photon must be equal to the energy difference between the two states.

SET UP and **EXECUTE:** The second excited state energy is $E_3 = \frac{9\pi^2\hbar^2}{2mL^2}$. The ground state energy is $E_1 = \frac{\pi^2\hbar^2}{2mL^2}$. $E_1 = 1.00$ eV, so $E_3 = 9.00$ eV. For the transition $\Delta E = \frac{4\pi^2\hbar^2}{mL^2}$. $\frac{hc}{\lambda} = \Delta E$.

$$\lambda = \frac{hc}{\Delta E} = \frac{(4.136\times10^{-15}\ \text{eV}\cdot\text{s})(2.998\times10^8\ \text{m/s})}{8.00\ \text{eV}} = 1.55\times10^{-7}\ \text{m} = 155\ \text{nm}.$$

EVALUATE: This wavelength is much shorter than those of visible light.

40.17. **IDENTIFY:** If the given wave function is a solution to the Schrödinger equation, we will get an identity when we substitute that wave function into the Schrödinger equation.

SET UP: We must substitute the equation $\Psi(x,t) = \sqrt{\frac{2}{L}}\sin\left(\frac{n\pi x}{L}\right)e^{-iE_n t/\hbar}$ into the one-dimensional Schrödinger equation $-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + U(x)\psi(x) = E\psi(x)$.

EXECUTE: Taking the second derivative of $\Psi(x,t)$ with respect to x gives $\frac{d^2\Psi(x,t)}{dx^2} = -\left(\frac{n\pi}{L}\right)^2\Psi(x,t)$.

Substituting this result into $-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + U(x)\psi(x) = E\psi(x)$, we get $\frac{\hbar^2}{2m}\left(\frac{n\pi}{L}\right)^2\Psi(x,t) = E\Psi(x,t)$ which gives $E_n = \frac{\hbar^2}{2m}\left(\frac{n\pi}{L}\right)^2$, the energies of a particle in a box.

EVALUATE: Since this process gives us the energies of a particle in a box, the given wave function is a solution to the Schrödinger equation

40.18. **IDENTIFY:** Find x where ψ_1 is zero and where it is a maximum.

SET UP: $\psi_1 = \sqrt{\frac{2}{L}}\sin\left(\frac{\pi x}{L}\right)$.

EXECUTE: **(a)** The wave function for $n = 1$ vanishes only at $x = 0$ and $x = L$ in the range $0 \le x \le L$.
(b) In the range for x, the sine term is a maximum only at the middle of the box, $x = L/2$.

EVALUATE: **(c)** The answers to parts (a) and (b) are consistent with the figure.

40.19. **IDENTIFY** and **SET UP:** For the $n = 2$ first excited state the normalized wave function is given by Eq. (40.35). $\psi_2(x) = \sqrt{\frac{2}{L}}\sin\left(\frac{2\pi x}{L}\right)$. $|\psi_2(x)|^2\,dx = \frac{2}{L}\sin^2\left(\frac{2\pi x}{L}\right)dx$. Examine $|\psi_2(x)|^2\,dx$ and find where it is zero and where it is maximum.

EXECUTE: **(a)** $|\psi_2|^2\,dx = 0$ implies $\sin\left(\frac{2\pi x}{L}\right) = 0$

$\frac{2\pi x}{L} = m\pi$, $m = 0, 1, 2, \ldots$; $x = m(L/2)$

For $m = 0$, $x = 0$; for $m = 1$, $x = L/2$; for $m = 2$, $x = L$

The probability of finding the particle is zero at $x = 0$, $L/2$, and L.

(b) $|\psi_2|^2\,dx$ is maximum when $\sin\left(\frac{2\pi x}{L}\right)=\pm 1$

$\frac{2\pi x}{L}=m(\pi/2),\ m=1,3,5,\ldots;\ x=m(L/4)$

For $m=1,\ x=L/4$; for $m=3,\ x=3L/4$

The probability of finding the particle is largest at $x=L/4$ and $3L/4$.

(c) EVALUATE: The answers to part (a) correspond to the zeros of $|\psi|^2$ shown in Figure 40.12 in the textbook and the answers to part (b) correspond to the two values of x where $|\psi|^2$ in the figure is maximum.

40.20. **IDENTIFY:** Evaluate $\frac{d^2\psi}{dx^2}$ and see if Eq. (40.25) is satisfied. $\psi(x)$ must be zero at the walls, where $U\to\infty$.

SET UP: $\frac{d}{dx}\sin kx=k\cos kx.\ \frac{d}{dx}\cos kx=-k\sin kx.$

EXECUTE: **(a)** $\frac{d^2\psi}{dx^2}=-k^2\psi$, and for ψ to be a solution of Eq. (40.25), $k^2=E\frac{2m}{\hbar^2}$.

(b) The wave function must vanish at the rigid walls; the given function will vanish at $x=0$ for any k, but to vanish at $x=L,\ kL=n\pi$ for integer n.

EVALUATE: From Eq. (40.31), $E_n=\frac{n^2\pi^2\hbar^2}{2mL^2}$, so $k_n=\frac{n\pi}{L}$ and $\psi=A\sin kx$ is the same as ψ_n in Eq. (40.32), except for a different symbol for the normalization constant

40.21. **(a) IDENTIFY and SET UP:** $\psi=A\cos kx$. Calculate $d\psi^2/dx^2$ and substitute into Eq. (40.25) to see if this equation is satisfied.

EXECUTE: Eq. (40.25): $-\frac{h^2}{8\pi^2 m}\frac{d^2\psi}{dx^2}=E\psi$

$$\frac{d\psi}{dx}=A(-k\sin kx)=-Ak\sin kx$$

$$\frac{d^2\psi}{dx^2}=-Ak(k\cos kx)=-Ak^2\cos kx$$

Thus Eq. (40.25) requires $-\frac{h^2}{8\pi^2 m}(-Ak^2\cos kx)=E(A\cos kx)$.

This says $\frac{h^2k^2}{8\pi^2 m}=E;\ k=\frac{\sqrt{2mE}}{(h/2\pi)}=\frac{\sqrt{2mE}}{\hbar}$

$\psi=A\cos kx$ is a solution to Eq. (40.25) if $k=\frac{\sqrt{2mE}}{\hbar}$.

(b) EVALUATE: The wave function for a particle in a box with rigid walls at $x=0$ and $x=L$ must satisfy the boundary conditions $\psi=0$ at $x=0$ and $\psi=0$ at $x=L$. $\psi(0)=A\cos 0=A$, since $\cos 0=1$. Thus ψ is not 0 at $x=0$ and this wave function isn't acceptable because it doesn't satisfy the required boundary condition, even though it is a solution to the Schrödinger equation.

40.22. **IDENTIFY:** The energy levels are given by Eq. (40.31). The wavelength λ of the photon absorbed in an atomic transition is related to the transition energy ΔE by $\lambda=\frac{hc}{\Delta E}$.

SET UP: For the ground state $n=1$ and for the third excited state $n=4$.

EXECUTE: **(a)** The third excited state is $n=4$, so

$$\Delta E=(4^2-1)\frac{h^2}{8mL^2}=\frac{15(6.626\times10^{-34}\ \text{J}\cdot\text{s})^2}{8(9.11\times10^{-31}\ \text{kg})(0.125\times10^{-9}\ \text{m})^2}=5.78\times10^{-17}\ \text{J}=361\ \text{eV}.$$

(b) $\lambda = \dfrac{hc}{\Delta E} = \dfrac{(6.63\times10^{-34}\text{ J}\cdot\text{s})(3.0\times10^{8}\text{ m/s})}{5.78\times10^{-17}\text{ J}} = 3.44\text{ nm}$

EVALUATE: This photon is an x ray. As the width of the box increases the transition energy for this transition decreases and the wavelength of the photon increases.

40.23. **IDENTIFY** and **SET UP:** $\lambda = \dfrac{h}{p} = \dfrac{h}{\sqrt{2mE}}$. The energy of the electron in level n is given by Eq. (40.31).

EXECUTE: **(a)** $E_1 = \dfrac{h^2}{8mL^2} \Rightarrow \lambda_1 = \dfrac{h}{\sqrt{2mh^2/8mL^2}} = 2L = 2(3.0\times10^{-10}\text{ m}) = 6.0\times10^{-10}\text{ m}$. The wavelength is twice the width of the box. $p_1 = \dfrac{h}{\lambda_1} = \dfrac{(6.63\times10^{-34}\text{ J}\cdot\text{s})}{6.0\times10^{-10}\text{ m}} = 1.1\times10^{-24}\text{ kg}\cdot\text{m/s}$.

(b) $E_2 = \dfrac{4h^2}{8mL^2} \Rightarrow \lambda_2 = L = 3.0\times10^{-10}\text{ m}$. The wavelength is the same as the width of the box.

$p_2 = \dfrac{h}{\lambda_2} = 2p_1 = 2.2\times10^{-24}\text{ kg}\cdot\text{m/s}$.

(c) $E_3 = \dfrac{9h^2}{8mL^2} \Rightarrow \lambda_3 = \dfrac{2}{3}L = 2.0\times10^{-10}\text{ m}$. The wavelength is two-thirds the width of the box.

$p_3 = 3p_1 = 3.3\times10^{-24}\text{ kg}\cdot\text{m/s}$.

EVALUATE: In each case the wavelength is an integer multiple of $\lambda/2$. In the n^{th} state, $p_n = np_1$.

40.24. **IDENTIFY:** To describe a real situation, a wave function must be normalizable.

SET UP: $|\psi|^2\, dV$ is the probability that the particle is found in volume dV. Since the particle must be *somewhere*, ψ must have the property that $\int|\psi|^2 dV = 1$ when the integral is taken over all space.

EXECUTE: **(a)** In one dimension, as we have here, the integral discussed above is of the form $\int_{-\infty}^{\infty}|\psi(x)|^2 dx = 1$.

(b) Using the result from part (a), we have $\int_{-\infty}^{\infty}(e^{ax})^2\, dx = \int_{-\infty}^{\infty} e^{2ax} dx = \left.\dfrac{e^{2ax}}{2a}\right|_{-\infty}^{\infty} = \infty$. Hence this wave function cannot be normalized and therefore cannot be a valid wave function.

(c) We only need to integrate this wave function of 0 to ∞ because it is zero for $x<0$. For normalization we have $1 = \int_{-\infty}^{\infty}|\psi|^2 dx = \int_0^{\infty}(Ae^{-bx})^2\, dx = \int_0^{\infty} A^2 e^{-2bx} dx = \left.\dfrac{A^2 e^{-2bx}}{-2b}\right|_0^{\infty} = \dfrac{A^2}{2b}$, which gives $\dfrac{A^2}{2b} = 1$, so $A = \sqrt{2b}$.

EVALUATE: If b were negative, the given wave function could not be normalized, so it would not be allowable.

40.25. **IDENTIFY:** Compare $-\dfrac{\hbar^2}{2m}\dfrac{d^2\psi}{dx^2} + U\psi$ to $E\psi$ and see if there is a value of k for which they are equal.

SET UP: $\dfrac{d^2}{dx^2}\sin kx = -k^2\sin kx$.

EXECUTE: **(a)** Eq. (40.23): $\dfrac{-\hbar^2}{2m}\dfrac{d^2\psi}{dx^2} + U\psi = E\psi$.

Left-hand side: $\dfrac{-\hbar^2}{2m}\dfrac{d^2}{dx^2}(A\sin kx) + U_0 A\sin kx = \dfrac{\hbar^2k^2}{2m}A\sin kx + U_0 A\sin kx = \left(\dfrac{\hbar^2k^2}{2m} + U_0\right)\psi$. But $\dfrac{\hbar^2k^2}{2m} + U_0 > U_0 > E$ if k is real. But $\dfrac{\hbar^2k^2}{2m} + U_0$ should equal E. This is not the case, and there is no k for which this $|\psi|^2$ is a solution.

(b) If $E > U_0$, then $\dfrac{\hbar^2 k^2}{2m} + U_0 = E$ is consistent and so $\psi = A\sin kx$ is a solution of Eq. (40.23) for this case.

EVALUATE: For a square-well potential and $E < U_0$, Eq. (40.23) with $U = U_0$ applies outside the well and the wave function has the form of Eq. (40.40).

40.26. **IDENTIFY:** $\lambda = \dfrac{h}{p}$. p is related to E by $E = \dfrac{p^2}{2m} + U$.

SET UP: For $x > L$, $U = U_0$. For $0 < x < L$, $U = 0$.

EXECUTE: For $0 < x < L$, $p = \sqrt{2mE} = \sqrt{2m(3U_0)}$ and $\lambda_{\text{in}} = \dfrac{h}{\sqrt{2m(3U_0)}}$. For $x > L$,

$p = \sqrt{2m(E - U_0)} = \sqrt{2m(2U_0)}$ and $\lambda_{\text{out}} = \dfrac{h}{\sqrt{2m(E-U_0)}} = \dfrac{h}{\sqrt{2m(2U_0)}}$. Thus, the ratio of the

wavelengths is $\dfrac{\lambda_{\text{out}}}{\lambda_{\text{in}}} = \dfrac{\sqrt{2m(3U_0)}}{\sqrt{2m(2U_0)}} = \sqrt{\dfrac{3}{2}}$.

EVALUATE: For $x > L$ some of the energy is potential and the kinetic energy is less than it is for $0 < x < L$, where $U = 0$. Therefore, outside the box p is less and λ is greater than inside the box.

40.27. **IDENTIFY:** Figure 40.15b in the textbook gives values for the bound state energy of a square well for which $U_0 = 6E_{\text{1-1DW}}$.

SET UP: $E_{\text{1-1DW}} = \dfrac{\pi^2 \hbar^2}{2mL^2}$.

EXECUTE: $E_1 = 0.625 E_{\text{1-1DW}} = 0.625 \dfrac{\pi^2\hbar^2}{2mL^2}$; $E_1 = 2.00\ \text{eV} = 3.20 \times 10^{-19}$ J.

$$L = \pi\hbar \left(\frac{0.625}{2(9.109 \times 10^{-31}\ \text{kg})(3.20 \times 10^{-19}\ \text{J})} \right)^{1/2} = 3.43 \times 10^{-10}\ \text{m}.$$

EVALUATE: As L increases the ground state energy decreases.

40.28. **IDENTIFY:** The energy of the photon is the energy given to the electron.

SET UP: Since $U_0 = 6E_{\text{1-1DW}}$ we can use the result $E_1 = 0.625 E_{\text{1-1DW}}$ from Section 40.4. When the electron is outside the well it has potential energy U_0, so the minimum energy that must be given to the electron is $U_0 - E_1 = 5.375 E_{\text{1-1DW}}$.

EXECUTE: The maximum wavelength of the photon would be

$$\lambda = \frac{hc}{U_0 - E_1} = \frac{hc}{(5.375)(h^2/8mL^2)} = \frac{8mL^2 c}{(5.375)h} = \frac{8(9.11 \times 10^{-31}\ \text{kg})(1.50 \times 10^{-9}\ \text{m})^2 (3.00 \times 10^8\ \text{m/s})}{(5.375)(6.63 \times 10^{-34}\ \text{J}\cdot\text{s})}$$
$$= 1.38 \times 10^{-6}\ \text{m}$$

EVALUATE: This photon is in the infrared. The wavelength of the photon decreases when the width of the well decreases.

40.29. **IDENTIFY:** Calculate $\dfrac{d^2\psi}{dx^2}$ and compare to $-\dfrac{2mE}{\hbar^2}\psi$.

SET UP: $\dfrac{d}{dx}\sin kx = k\cos kx$. $\dfrac{d}{dx}\cos kx = -k\sin kx$.

EXECUTE: Eq. (40.37): $\psi = A\sin\dfrac{\sqrt{2mE}}{\hbar}x + B\cos\dfrac{\sqrt{2mE}}{\hbar}x$.

$\dfrac{d^2\psi}{dx^2} = -A\left(\dfrac{2mE}{\hbar^2}\right)\sin\dfrac{\sqrt{2mE}}{\hbar}x - B\left(\dfrac{2mE}{\hbar^2}\right)\cos\dfrac{\sqrt{2mE}}{\hbar}x = \dfrac{-2mE}{\hbar^2}(\psi)$. This is Eq. (40.38), so this ψ is a solution.

EVALUATE: ψ in Eq. (40.38) is a solution to Eq. (40.37) for any values of the constants A and B.

40.30. IDENTIFY: The longest wavelength corresponds to the smallest energy change.

SET UP: The ground level energy level of the infinite well is $E_{\text{1-1DW}} = \dfrac{h^2}{8mL^2}$, and the energy of the photon must be equal to the energy difference between the two shells.

EXECUTE: The 400.0 nm photon must correspond to the $n = 1$ to $n = 2$ transition. Since $U_0 = 6E_{\text{1-1DW}}$, we have $E_2 = 2.43E_{\text{1-1DW}}$ and $E_1 = 0.625E_{\text{1-1DW}}$. The energy of the photon is equal to the energy difference between the two levels, and $E_{\text{1-1DW}} = \dfrac{h^2}{8mL^2}$, which gives

$E_\gamma = E_2 - E_1 \Rightarrow \dfrac{hc}{\lambda} = (2.43 - 0.625)E_{\text{1-1DW}} = \dfrac{1.805\, h^2}{8mL^2}$. Solving for L gives

$$L = \sqrt{\frac{(1.805)h\lambda}{8mc}} = \sqrt{\frac{(1.805)(6.626\times10^{-34}\ \text{J}\cdot\text{s})(4.00\times10^{-7}\ \text{m})}{8(9.11\times10^{-31}\ \text{kg})(3.00\times10^{8}\ \text{m/s})}} = 4.68\times10^{-10}\ \text{m} = 0.468\ \text{nm}.$$

EVALUATE: This width is approximately half that of a Bohr hydrogen atom.

40.31. IDENTIFY: Find the transition energy ΔE and set it equal to the energy of the absorbed photon. Use $E = hc/\lambda$, to find the wavelength of the photon.

SET UP: $U_0 = 6E_{\text{1-1DW}}$, as in Figure 40.15 in the textbook, so $E_1 = 0.625E_{\text{1-1DW}}$ and $E_3 = 5.09E_{\text{1-1DW}}$ with $E_{\text{1-1DW}} = \dfrac{\pi^2\hbar^2}{2mL^2}$. In this problem the particle bound in the well is a proton, so $m = 1.673\times10^{-27}$ kg.

EXECUTE: $E_{\text{1-1DW}} = \dfrac{\pi^2\hbar^2}{2mL^2} = \dfrac{\pi^2(1.055\times10^{-34}\ \text{J}\cdot\text{s})^2}{2(1.673\times10^{-27}\ \text{kg})(4.0\times10^{-15}\ \text{m})^2} = 2.052\times10^{-12}$ J. The transition energy is $\Delta E = E_3 - E_1 = (5.09 - 0.625)E_{\text{1-1DW}} = 4.465E_{\text{1-1DW}}$. $\Delta E = 4.465(2.052\times10^{-12}\ \text{J}) = 9.162\times10^{-12}$ J

The wavelength of the photon that is absorbed is related to the transition energy by $\Delta E = hc/\lambda$, so

$$\lambda = \frac{hc}{\Delta E} = \frac{(6.626\times10^{-34}\ \text{J}\cdot\text{s})(2.998\times10^{8}\ \text{m/s})}{9.162\times10^{-12}\ \text{J}} = 2.2\times10^{-14}\ \text{m} = 22\ \text{fm}.$$

EVALUATE: The wavelength of the photon is comparable to the size of the box.

40.32. IDENTIFY: The tunneling probability is $T = Ge^{-2\kappa L}$, with $G = 16\dfrac{E}{U_0}\left(1 - \dfrac{E}{U_0}\right)$ and $\kappa = \dfrac{\sqrt{2m(U_0 - E)}}{\hbar}$. so

$$T = 16\frac{E}{U_0}\left(1 - \frac{E}{U_0}\right)e^{\frac{-2\sqrt{2m(U_0 - E)}}{\hbar}L}.$$

SET UP: $U_0 = 30.0\times10^6$ eV, $L = 2.0\times10^{-15}$ m, $m = 6.64\times10^{-27}$ kg.

EXECUTE: **(a)** $U_0 - E = 1.0\times10^6$ eV ($E = 29.0\times10^6$ eV), $T = 0.090$.

(b) If $U_0 - E = 10.0\times10^6$ eV ($E = 20.0\times10^6$ eV), $T = 0.014$.

EVALUATE: T is less when $U_0 - E$ s 10.0 MeV than when $U_0 - E$ is 1.0 MeV.

40.33. IDENTIFY: The tunneling probability is $T = 16\dfrac{E}{U_0}\left(1 - \dfrac{E}{U_0}\right)e^{-2L\sqrt{2m(U_0 - E)}/\hbar}$.

SET UP: $\dfrac{E}{U_0} = \dfrac{6.0\ \text{eV}}{11.0\ \text{eV}}$ and $E - U_0 = 5\ \text{eV} = 8.0\times10^{-19}$ J.

EXECUTE: **(a)** $L = 0.80\times10^{-9}$ m:

$$T = 16\left(\frac{6.0\ \text{eV}}{11.0\ \text{eV}}\right)\left(1 - \frac{6.0\ \text{ev}}{11.0\ \text{eV}}\right)e^{-2(0.80\times10^{-9}\ \text{m})\sqrt{2(9.11\times10^{-31}\ \text{kg})(8.0\times10^{-19}\ \text{J})}/1.055\times10^{-34}\ \text{J}\cdot\text{s}} = 4.4\times10^{-8}.$$

(b) $L = 0.40\times10^{-9}$ m: $T = 4.2\times10^{-4}$.

EVALUATE: The tunneling probability is less when the barrier is wider.