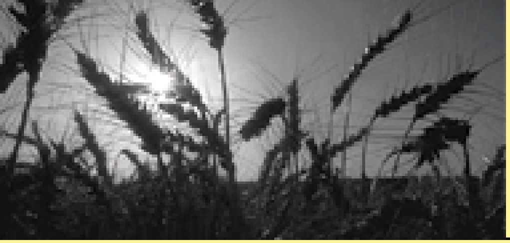
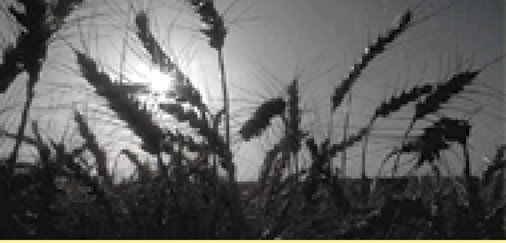


第四节 差分与等距节点newton插值

- 5.5.1、差分及其性质
- 5.5.2、等距节点插值公式
- 5.5.3、例题分析



一、差分及其性质

在实际应用Newton插值多项式时，经常碰到插值节点是等距的状况，此时能够简化Newton插值公式。

当插值节点$x_0,x_1,\ldots,x_n$分布等距时，也即

$$h=x_{k+1}-x_k,\quad k=0,1,2,\ldots,n-1$$

已知$n+1$个插值节点：

$x_k=x_0+kh\ \ (k=0,1,L\ ,n)$ 其中$h=\dfrac{x_n-x_0}{n}$ 为步长

于是在差商中，分母部分将变得简单，

计算量主要集中在分子（两节点处函数值的差）。

分析差商的形式，引入差分概念

定义5.5.1. 设$f(x)$在等距节点$x_k = x_0 + kh$处的函数值为f_k，

$k = 0,1,\cdots,n$，称

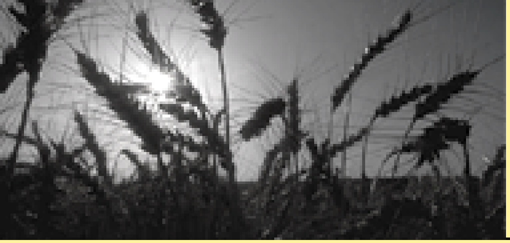
$\Delta f_k = f_{k+1} - f_k \quad k = 0,1,\cdots,n-1$ 为$f(x)$在x_k处的一阶向前差分

$\nabla f_k = f_k - f_{k-1} \quad k = 1,2,\cdots,n$ 为$f(x)$在x_k处的一阶向后差分

$$\delta f(x_i) = f(x_i + \frac{h}{2}) - f(x_i - \frac{h}{2})$$ 一阶中心差分

$\Delta^2 f_k = \Delta f_{k+1} - \Delta f_k$ 为$f(x)$在x_k处的二阶向前差分

$\nabla^2 f_k = \nabla f_k - \nabla f_{k-1}$ 为$f(x)$在x_k处的二阶向后差分

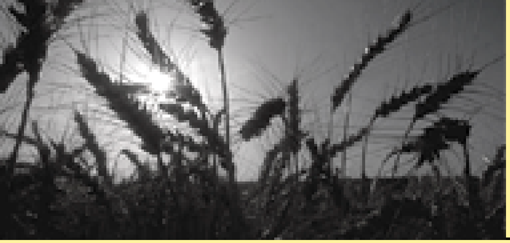


依这类推

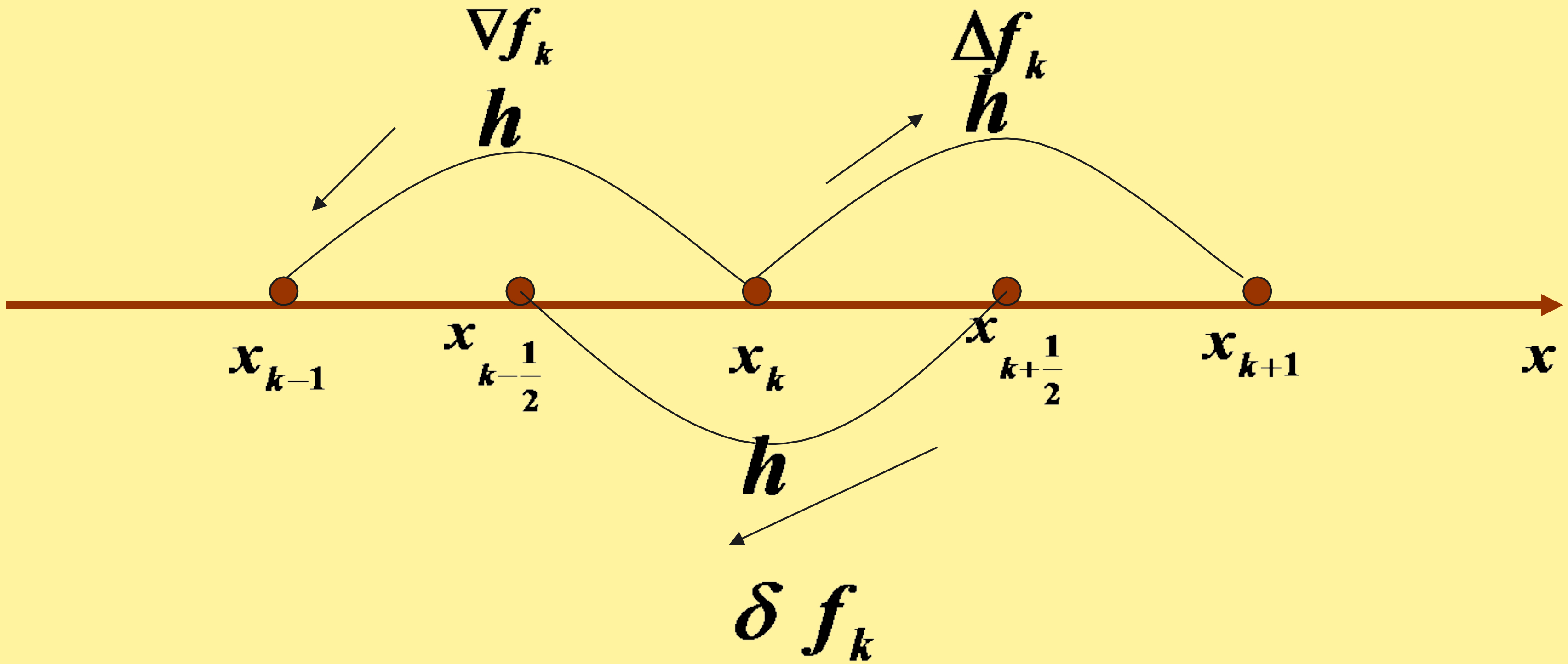
$$\Delta^m f_k = \Delta^{m-1} f_{k+1} - \Delta^{m-1} f_k$$ 为$f(x)$在x_k处的m阶向前差分

$$\nabla^m f_k = \nabla^{m-1} f_k - \nabla^{m-1} f_{k-1}$$ 为$f(x)$在x_k处的m阶向后差分

$$\delta^m f_k = \delta^{m-1} f_{k+1/2} - \delta^{m-1} f_{k-1/2}$$



差 分



符号Δ，∇，δ 分别称为**向前差分算子**，向后差分算子及中心差分算子

引入下列惯用算子符号：

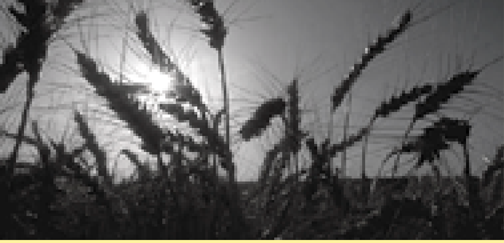
$$If_k = f_k \qquad Ef_k = f_{k+1}$$

并称I为恒等算子，E为移位算子，各算子之间以下关系

$$\Delta f_k = f_{k+1} - f_k = Ef_k - If_k = (E - I) f_k$$

故 $\Delta = (E - I)$ 同理

$$\delta = E^{\frac{1}{2}} - E^{-\frac{1}{2}} \qquad \nabla = (I - E^{-1})$$

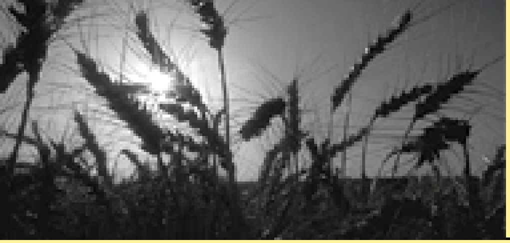


差分的性质

性质5.5　常数的差分等于零

性质5.6　函数值能够表达各阶差分

$$
\begin{aligned}
\Delta^n f_k &= (E - I)^n f_k \\
&= \sum_{j=0}^{n} (-1)^j C_n^j E^{n-j} f_k \\
&= \sum_{j=0}^{n} (-1)^j C_n^j f_{n+k-j}
\end{aligned}
$$



$$\nabla^n f_k = (I - E^{-1})^n f_k$$

$$= \sum_{j=0}^{n} (-1)^{n-j} C_n^j E^{j-n} f_k$$

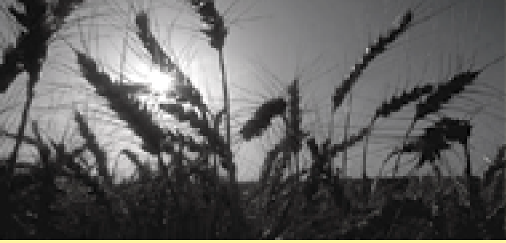
$$= \sum_{j=0}^{n} (-1)^{n-j} C_n^j f_{k+j-n}$$

性质5.7 各阶差分可以表示函数值.

$$f_{n+k} = E^n f_k = (I + \Delta)^n f_k$$

$$= (\sum_{j=0}^{n} C_n^{\ j} \Delta^j) f_k.$$

$$f_{n+k} = E^n f_k = \sum_{j=0}^{n} C_n^{\ j} \Delta^j f_k.$$



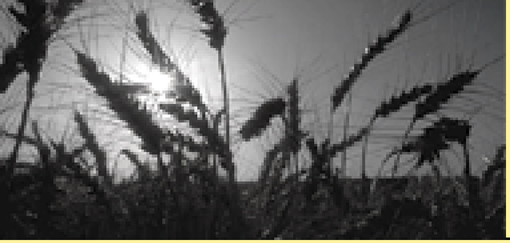
性质5.8 均差与差分的关系，

例如，对向前差分，由定义

$$f[x_k, x_{k+1}] = \frac{f_{k+1} - f_k}{x_{k+1} - x_k} = \frac{\Delta f_k}{h} \quad ,$$

$$f[x_k, x_{k+1}, x_{k+2}] = \frac{f[x_{k+1}, x_{k+2}] - f[x_k, x_{k+1}]}{x_{k+2} - x_k}$$

$$= \frac{\dfrac{\Delta f_{k+1}}{h} - \dfrac{\Delta f_k}{h}}{2h} = \frac{1}{2h^2}\Delta^2 f_k ,$$



一般地有 $f[x_k,\cdots,x_{k+m}]=\dfrac{1}{m!}\dfrac{1}{h^m}\Delta^m f_k$

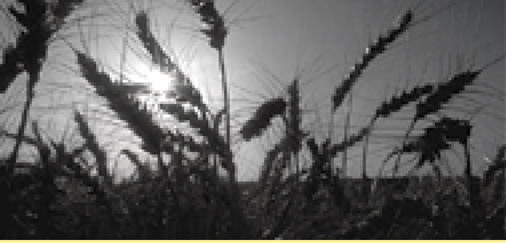
$$(m=1,2,\cdots,n). \quad (4.7)$$

同理，对向后差分有

$$f[x_k,x_{k-1},\cdots,x_{k-m}]=\frac{1}{m!}\frac{1}{h^m}\nabla^m f_k$$

差分与导数的关系：

$$\because f[x_0,\cdots,x_n]=\frac{f^{(n)}(\xi)}{n!}, \qquad \xi\in[a,b].$$



又 $f[x_k, \mathrm{K}\ , x_{k+n}] = \dfrac{1}{n!}\dfrac{1}{h^n}\Delta^n f_k$

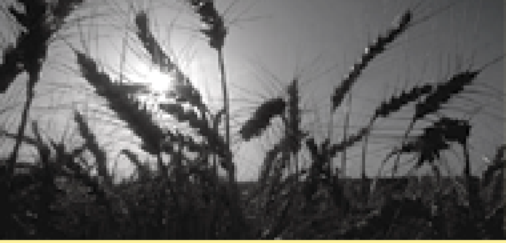
$$f[x_k, \mathrm{K}\ , x_{k+n}] = \frac{f^{(n)}(\xi)}{n!}$$

$$\therefore \quad \Delta^n f_k = h^n f^{(n)}(\xi),$$

$$\therefore \quad f^{(n)}(\xi) = \frac{\Delta^n f_k}{h^n}$$

差分表

x_k	f_k	一阶差分	二阶差分	三阶差分	四阶差分
x_0	f_0				
		Δf_0 ∇f_1			
x_1	f_1		$\Delta^2 f_0$ $\nabla^2 f_2$		
		Δf_1 ∇f_2		$\Delta^3 f_0$ $\nabla^3 f_3$	
x_2	f_2		$\Delta^2 f_1$ $\nabla^2 f_3$		$\Delta^4 f_0$ $\nabla^4 f_4$
		Δf_2 ∇f_3		$\Delta^3 f_1$ $\nabla^3 f_4$	
x_3	f_3		$\Delta^2 f_2$ $\nabla^2 f_4$		
		Δf_3 ∇f_4			
x_4	f_4				



5.5.2、Newton插值公式

1.Newton向前(差分)插值公式

如果节点$x_0, x_1, \cdots, x_n$是等距节点即

$$x_k = x_0 + kh, k = 0, 1, \cdots, n, h = \frac{b-a}{n}$$

Newton插值基本公式为

$$N_n(x) = f_0 + \sum_{k=1}^{n} f[x_0, x_1, \cdots, x_k]\omega_k(x)$$

如果假设 $x = x_0 + th$ 计算x_0点附近的值

由差商与向前差分的关系 $f[x_0, x_1, \cdots, x_k] = \dfrac{\Delta^k f_0}{k! \cdot h^k}$