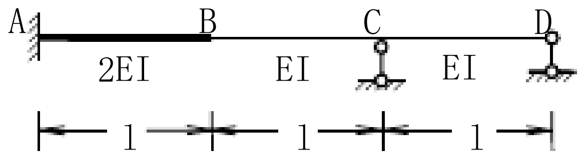
同济大学朱慈勉 结构力学 第 8 章 矩阵位移法习题答案

8-1 试说出单元刚度矩阵的物理意义及其性质与特点。

8-2 试说出空间桁架和刚架单元刚度矩阵的阶数。

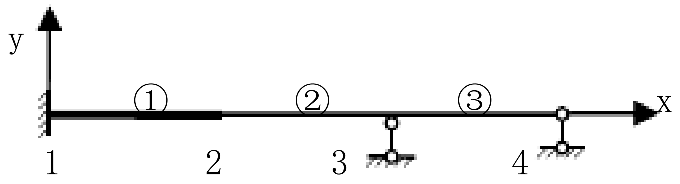
8-3 试分别采用后处理法和先处理法列出图示梁的结构刚度矩阵。

(a)



解：（a）用后处理法计算

（1）结构标识



单元	局部坐标系（i→ j)	杆长	cosα	sinα	各杆 EI
①	1 → 2	l	1	0	2EI
②	2 → 3	l	1	0	EI
③	3 → 4	l	1	0	EI

（2）建立结点位移向量，结点力向量

$$= \left[v_1 \ \ \theta_1 \ \ v_2 \ \ \theta_2 \ \ v_3 \ \ \theta_3 \ \ v_4 \ \ \theta_4 \right]^T$$

$$F_\theta = \left[F_{y1} \ \ M_1 \ \ F_{y2} \ \ M_2 \ \ F_{y3} \ \ M_3 \ \ F_{y4} \ \ M_4 \right]^T$$

（3）计算单元刚度矩阵

$$k^{①} = \begin{bmatrix} k_{11}^{①} & k_{12}^{①} \\ k_{21}^{①} & k_{22}^{①} \end{bmatrix} = \frac{2EI}{l^3} \begin{array}{cccc} \multicolumn{2}{c}{1} & \multicolumn{2}{c}{2} \\ 12 & 6l & 12- & 6l \\ 6l & 4l^2 & 6{-}l & 2l^2 \\ 12 & 6{-}l & 12 & 6{-}l \\ 6l & 2l^2 & 6{-}l & 4l^2 \end{array}$$

$$k^{②} = \begin{bmatrix} k_{22}^{②} & k_{23}^{②} \\ k_{32}^{②} & k_{33}^{②} \end{bmatrix} = \frac{2EI}{l^3} \begin{array}{cccc} \multicolumn{2}{c}{1} & \multicolumn{2}{c}{2} \\ 6 & 3l & -6 & 3l \\ 3l & 2l^2 & 3{-}l & {}_2 \ l \\ 6 & 3{-}l & 6 & -3l \\ 3l & {}_2 & l-3l & 2l^2 \end{array}$$

$$k^{③} = \begin{bmatrix} k_{33}^{③} & k_{34}^{③} \\ k_{43}^{③} & k_{44}^{③} \end{bmatrix} = \frac{2EI}{l^3}\begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \begin{bmatrix} 6 & 3l & -6 & 3l \\ 3l & 2l^2 & -3l & l^2 \\ 6 & -3l & 6 & -3l \\ 3l & l^2 & -3l & 2l^2 \end{bmatrix} \end{array}$$

（4）总刚度矩阵

$$k_{\theta} = \begin{array}{cccc} 1 & 2 & 3 & 4 \\ \hline k_{11}^{①} & k_{12}^{①} & 0 & 0 \\ k_{21}^{①} & k_{22}^{①}+k_{22}^{②} & k_{23}^{②} & 0 \\ 0 & k_{32}^{②} & k_{33}^{②}+k_{33}^{③} & k_{34}^{③} \\ 0 & 0 & k_{43}^{③} & k_{44}^{③} \end{array} = \frac{2EI}{l^3}\begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 12 & 6l & -12 & 6l & 0 & 0 & 0 & 0 \\ 6l & 4l^2 & -6l & 2l^2 & 0 & 0 & 0 & 0 \\ -12 & -6l & 18 & -3l & -6 & 3l & 0 & 0 \\ 6l & 2l^2 & -3l & 6l^2 & -3l & l^2 & 0 & 0 \\ 0 & 0 & 6 & -3l & 12 & 0 & 6 & -3l \\ 0 & 0 & 3l & l^2 & 0 & 4l^2 & -3l & l^2 \\ 0 & 0 & 0 & 0 & -6 & -3l & 6 & -3l \\ 0 & 0 & 0 & 0 & 3l & l^2 & -3l & 2l^2 \end{bmatrix} \end{array}$$

（5）建立结构刚度矩阵

支座位移边界条件

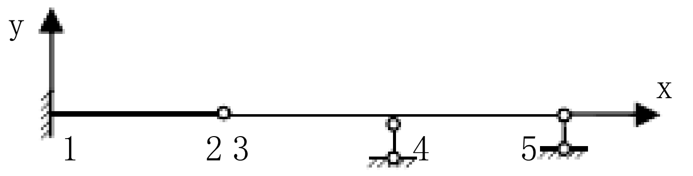
$$\begin{bmatrix} v_1 & \theta_1 & \theta_3 & \theta_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}$$

将总刚度矩阵中对应上述边界位移行列删除，得刚度结构矩阵。

$$k_{\theta} = \frac{2EI}{l^3}\begin{bmatrix} 18 & -3l & 3l & 0 \\ -3l & 6l^2 & l^2 & 0 \\ 3l & l^2 & 4l^2 & l^2 \\ 0 & 0 & l^2 & 2l^2 \end{bmatrix}$$

(b)用先处理法计算

（1）结构标识



单元	局部坐标系（i → j)	杆长	cosα	sinα	各杆 EI
①	1 → 2	l	0	1	2EI
②	2 → 3	l	0	1	EI
③	3 → 4	l	0	1	EI

（2）建立结点位移向量，结点力向量

$$= \begin{bmatrix} v_1 & \theta_1 & v_4 & v_5 \end{bmatrix}^T = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}^T$$

故 $= \begin{bmatrix} v_2 & \theta_2 & \theta_3 & \theta_4 & \theta_5 \end{bmatrix}^T$

（3）计算单元刚度矩阵

$$k① = \frac{2EI}{l^3}\begin{array}{cc} v_2 & \theta_2 \\ 12 & 6l - \\ -6l & 4l^2 \end{array}$$

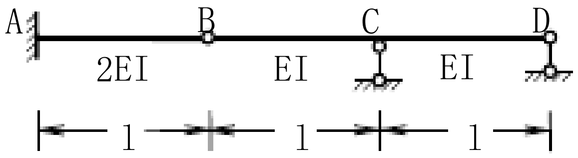
$$k② = \frac{EI}{l^3}\begin{array}{ccc} v_2 & \theta_3 & \theta_4 \\ 12 & 6l & 6l \\ 6l & 4l^2 & 2l^2 \\ 6l & 2l^2 & 4l^2 \end{array}$$

$$k③ = \frac{EI}{l^3}\begin{array}{cc} \theta_4 & \theta_5 \\ 4l^2 & {}_2 l \\ l^2 & 4l^2 \end{array}$$

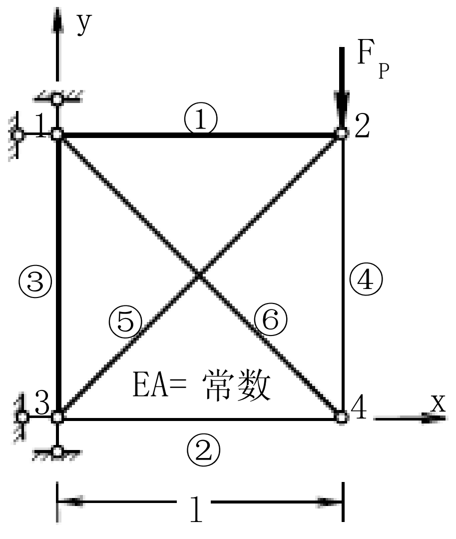
（4）建立结构刚度矩阵（按对号入座的方法）

$$k_\theta = \frac{2EI}{l^3}\begin{array}{ccccc} v_2 & \theta_2 & \theta_3 & \theta_4 & \theta_5 \\ 18 & 6l & -3l & 3l & 0 \\ -6l & 4l^2 & 0 & 0 & 0 \\ 3l & 0 & 2l^2 & {}_2\ l & 0 \\ 3l & 0 & l^2 & 4l^2 & l^2 \\ 0 & 0 & 0 & {}_2 & l2l^2 \end{array}$$

(b)



8-4 试分别采用后处理法和先处理法分析图示桁架，并将内力表示在图上。设各杆的EA 相同。



解：（1）结构标识如图

单元	局部坐标系（ $i\rightarrow j$）	杆长	cosα	sinα
①	$1\rightarrow 2$	l	1	0

②	3 → 4	1	1	0
③	1 → 3	1	0	−1
④	2 → 4	1	0	−1
⑤	2 → 3	$\sqrt{2}1$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$
⑥	1 → 4	$\sqrt{2}1$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$

（2）建立结点位移向量，结点力向量

$$= \left[\mu_1 \quad v_1 \quad \mu_2 \quad v_2 \quad \mu_3 \quad v_3 \quad \mu_4 \quad v_4\right]^T$$

$$F = \left[F_{x1} \quad F_{y1} \quad 0 \quad -F_p \quad F_{x3} \quad F_{y3} \quad 0 \quad 0\right]^T$$

（3）计算单元刚度矩阵

$$k\text{①} = \frac{EA}{1}\begin{array}{cccc} \multicolumn{2}{c}{1} & \multicolumn{2}{c}{2} \\ 1 & 0 & 1 & -0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}$$

同理

$$k\text{②} = k\text{①} = \frac{EA}{1}\begin{array}{cccc} \multicolumn{2}{c}{3} & \multicolumn{2}{c}{4} \\ 1 & 0 & 1 & -0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}$$

$$k\text{③} = \frac{EA}{1}\begin{array}{cccc} \multicolumn{2}{c}{1} & \multicolumn{2}{c}{3} \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{array}$$

同理

$$k\text{④} = k\text{③} = \frac{EA}{1}\begin{array}{cccc} \multicolumn{2}{c}{2} & \multicolumn{2}{c}{4} \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{array}$$

$$k\text{⑤} = \frac{EA}{\sqrt{2}1}\begin{array}{cccc} \multicolumn{2}{c}{2} & \multicolumn{2}{c}{3} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{array}$$

同理

$$k\text{⑥} = k\text{⑤} = \frac{EA}{\sqrt{2}1}\begin{array}{cccc} \multicolumn{2}{c}{1} & \multicolumn{2}{c}{4} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{array}$$

（4）形成刚度矩阵，刚度方程

$$k_0 = \frac{EA}{1}\begin{array}{cccccccc} 1 & & 2 & & 3 & & 4 & \\ \frac{4+\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} & -1 & 0 & 0 & 0 & -\frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} \\ -\frac{\sqrt{2}}{4} & \frac{4+\sqrt{2}}{4} & 0 & 0 & 0 & -1 & \frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} \\ -1 & 0 & \frac{4+\sqrt{2}}{4} & \frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} & 0 & 0 \\ 0 & 0 & \frac{\sqrt{2}}{4} & \frac{4+\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} & 0 & -1 \\ 0 & 0 & -\frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} & \frac{4+\sqrt{2}}{4} & \frac{\sqrt{2}}{4} & -1 & 0 \\ 0 & -1 & -\frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} & \frac{4+\sqrt{2}}{4} & 0 & 0 \\ -\frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} & 0 & 0 & -1 & 0 & \frac{4+\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} \\ \frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} & 0 & -1 & 0 & 0 & -\frac{\sqrt{2}}{4} & \frac{4+\sqrt{2}}{4} \end{array}$$

刚架总刚度矩阵方程：

$$k\begin{bmatrix}\mu_1 & v_1 & \mu_2 & v_2 & \mu_3 & v_3 & \mu_4 & v_4\end{bmatrix}^T = \begin{bmatrix}F_{x1} & F_{y1} & 0 & -F_p & F_{x3} & F_{y3} & 0 & 0\end{bmatrix}^T$$

（5）建立结构刚度矩阵，结构刚度方程

制作位移边界条件为：

$$\begin{matrix} {}_1 \\ {}_2 \end{matrix} = \begin{matrix}\mu_1 \\ v_1 \\ \mu_3 \\ v_3\end{matrix} = \begin{matrix}0 \\ 0 \\ 0 \\ 0\end{matrix}$$

将刚度矩阵中对应上述边界位移的行、列删除，即得结构刚度矩阵，相应结构刚度方程为：

$$\frac{EA}{1}\begin{matrix} \frac{4+\sqrt{2}}{4} & \frac{\sqrt{2}}{4} & 0 & 0 \\ \frac{\sqrt{2}}{4} & \frac{4+\sqrt{2}}{4} & 0 & -1 \\ 0 & 0 & \frac{4+\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} \\ 0 & -1 & -\frac{\sqrt{2}}{4} & \frac{4+\sqrt{2}}{4} \end{matrix} \begin{matrix}\mu_2 \\ v_2 \\ \mu_4 \\ v_4\end{matrix} = \begin{matrix}0 \\ -F_p \\ 0 \\ 0\end{matrix}$$

（6）计算节点位移，得：

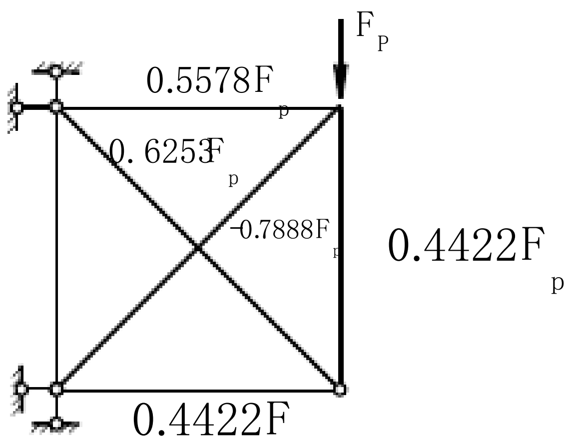
$$= \begin{bmatrix}\mu_2\\ v_2\\ \mu_4\\ v_4\end{bmatrix} = \frac{EA}{l}^{1} \begin{bmatrix} \frac{4+\sqrt{2}}{4} & \frac{\sqrt{2}}{4} & 0 & 0\\ \frac{\sqrt{2}}{4} & \frac{4+\sqrt{2}}{4} & 0 & -1 \\ 0 & 0 & \frac{4+\sqrt{2}}{4} & -\frac{\sqrt{2}}{4}\\ 0 & -1 & -\frac{\sqrt{2}}{4} & \frac{4+\sqrt{2}}{4}\end{bmatrix}^{1} \begin{bmatrix}0\\ -F_p\\ 0\\ 0\end{bmatrix} = \begin{bmatrix}0.5578\\ -2.1354\\ -0.4422\\ -1.6928\end{bmatrix}$$

（7）计算各杆内力

$$F^{(5)} = \frac{EA}{\sqrt{2}l} \begin{array}{cccc} \multicolumn{2}{c}{2} & \multicolumn{2}{c}{3} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2}\\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2}\\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2}\\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{array} \begin{bmatrix}0.5578\\ 2.1354\\ 0\\ 0\end{bmatrix} \frac{F_p l}{EA} = \begin{bmatrix}0.7888\\ 0.7888\\ 0.7888\\ 0.7888\end{bmatrix} \frac{F_p}{\sqrt{2}}$$

$$\overline{F}^{(5)} = T F^{(5)} = \frac{\sqrt{2}}{2} \begin{bmatrix} 1 & 1 & 0 & 0\\ -1 & 1 & 0 & 0\\ 0 & 0 & 1 & 1\\ 0 & 0 & -1 & 1\end{bmatrix} \begin{bmatrix}0.7888\\ 0.7888\\ 0.7888\\ 0.7888\end{bmatrix} \frac{F_p}{\sqrt{2}} = \begin{bmatrix}0.7888F_p\\ 0\\ 0.7888F_p\\ 0\end{bmatrix}$$

同时可得其他杆内力。



（b）采用先处理法

（1）步与后处理法相同。

（2）建立结点位移向量，结点力向量

$$= \begin{bmatrix}\mu_2 & v_2 & \mu_4 & v_4\end{bmatrix}^T$$

$$F = \begin{bmatrix}0 & F_p & 0 & 0\end{bmatrix}^T$$

$$k① = \frac{EA}{l}\begin{array}{cc} \multicolumn{2}{c}{2} \\ 1 & 0 \\ 0 & 0 \end{array} \qquad k② = k① = \frac{EA}{l}\begin{array}{cc} \multicolumn{2}{c}{4} \\ 1 & 0 \\ 0 & 0 \end{array}$$

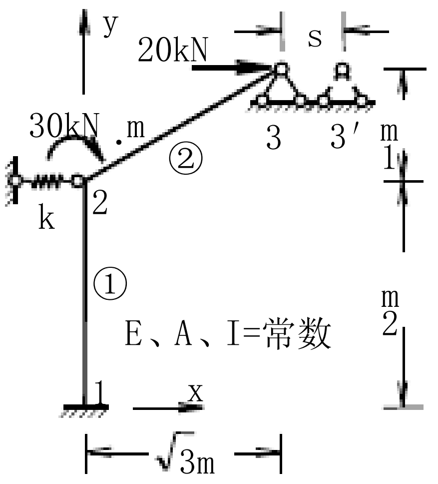
$$k④ = \frac{EA}{l}\begin{array}{cccc} \multicolumn{2}{c}{2} & \multicolumn{2}{c}{4} \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{array} \quad k⑤ = \frac{EA}{\sqrt{2}l}\begin{array}{cc} \multicolumn{2}{c}{4} \\ \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{array} \quad k⑥ = \frac{EA}{\sqrt{2}l}\begin{array}{cc} \multicolumn{2}{c}{4} \\ \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{array}$$

（4）形成总刚度矩阵，结构刚度方程

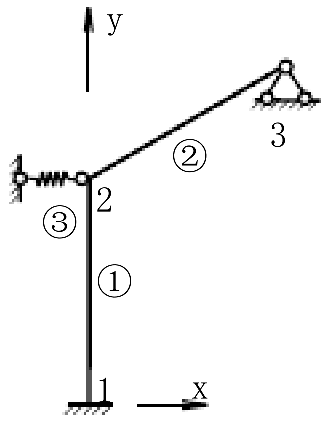
$$\frac{EA}{l}\begin{bmatrix} \frac{4+\sqrt{2}}{4} & \frac{\sqrt{2}}{4} & 0 & 0 \\ \frac{\sqrt{2}}{4} & \frac{4+\sqrt{2}}{4} & 0 & -1 \\ 0 & 0 & \frac{4+\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} \\ 0 & -1 & -\frac{\sqrt{2}}{4} & \frac{4+\sqrt{2}}{4} \end{bmatrix}\begin{bmatrix} \mu_2 \\ v_2 \\ \mu_4 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ -F_p \\ 0 \\ 0 \end{bmatrix}$$

（5）结点位移及内力计算同上。

8-5 试列出图示刚架的结构刚度方程。设杆件的 E、A、I均相同，结点 3 有水平支座位移 s，弹簧刚度系数为 k。



解：（1）结构标识



单元	局部坐标系（$i \rightarrow j$）	杆长	cosα	sinα
①	$1 \rightarrow 2$	2	0	1
②	$2 \rightarrow 3$	2	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$

（2）建立结点位移向量，结点力向量

$$= \begin{bmatrix} \mu_2 & v_2 & \theta_2 & \theta_3 \end{bmatrix}^T$$

$$F = \begin{bmatrix} 20 & 0 & -30 & 0 \end{bmatrix}^T$$

（3）建立单元刚度矩阵（1=2m）

$$k① = \begin{array}{ccc} \mu_2 & v_2 & \theta_2 \\ \frac{12EI}{l^3} & 0 & \frac{6EI}{l^3} \\ 0 & \frac{EA}{1} & 0 \\ \frac{6EI}{l^2} & 0 & \frac{4EI}{1} \end{array}$$

$$k② = \begin{array}{ccccc} \mu_2 & v_2 & \theta_2 & v_3 & \theta_3 \\ \frac{3EA}{41}+\frac{3EI}{l^3} & 0 & 0 & 0 & 0 \\ \frac{\sqrt{3}}{4}\ \frac{EA}{1}\ \frac{12EI}{l^3} & \frac{EA}{41}+\frac{9EI}{l^3} & 0 & 0 & 0 \\ \frac{3EI}{l^2} & \frac{3\sqrt{3}EI}{l^2} & \frac{4EI}{1} & 0 & 0 \\ \frac{3EA}{41}+\frac{3EI}{l^3} & -\frac{\sqrt{3}}{4}\ \frac{EA}{1}\ \frac{12EI}{l^3} & \frac{3EI}{l^2} & \frac{3EA}{41}+\frac{3EI}{l^3} & 0 \\ \frac{3EI}{l^2} & \frac{3\sqrt{3}EI}{l^2} & \frac{2EI}{1} & \frac{3EI}{l^2} & \frac{4EI}{1} \end{array}$$

$$k③ = k$$

（4）建立结构刚度方程（对号入座的原则写出保留支座位移 v_3 在内的刚度方程）

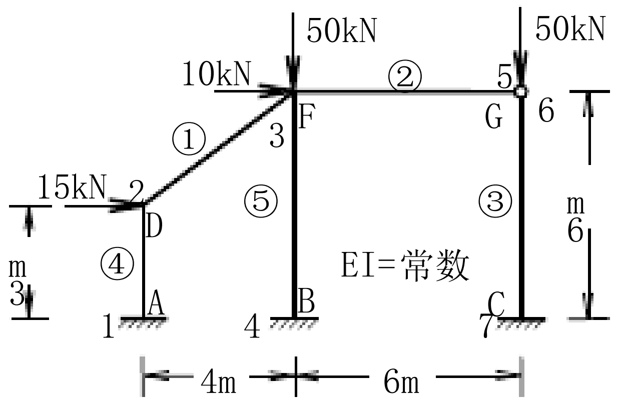
$$\begin{array}{ccccc} \frac{3EA}{41}+\frac{15EI}{l^3}+k & 0 & 0 & 0 & 0 \\ \frac{\sqrt{3}}{4}\ \frac{EA}{1}\ \frac{12EI}{l^3} & \frac{5EA}{41}+\frac{9EI}{l^3} & 0 & 0 & 0 \\ \frac{3EI}{l^2} & \frac{3\sqrt{3}EI}{l^2} & \frac{8EI}{1} & 0 & 0 \\ \frac{3EA}{41}+\frac{3EI}{l^3} & -\frac{\sqrt{3}}{4}\ \frac{EA}{1}\ \frac{12EI}{l^3} & \frac{3EI}{l^2} & \frac{3EA}{41}+\frac{3EI}{l^3} & 0 \\ \frac{3EI}{l^2} & \frac{3\sqrt{3}EI}{l^2} & \frac{2EI}{1} & \frac{3EI}{l^2} & \frac{4EI}{1} \end{array} \begin{array}{c} \mu_2 \\ v_2 \\ \theta_2 \\ v_3 \\ \theta_3 \end{array} = \begin{array}{c} 20KN \\ 0 \\ 30KN \\ F_{x3} \\ 0 \end{array}$$

由已知，支座位移 v_3 = c，将以上刚度矩阵 v_3 的行删除，并将 v_3 与刚度矩阵第 4 列乘

积移至方程右端与荷载向量合并。

$$\begin{bmatrix} \frac{3}{8}EA+\frac{15}{8}EI+k & \frac{\sqrt{3}}{4}\quad\frac{EA}{2}\quad\frac{3EI}{2} & \frac{3}{4}EI & \frac{3}{4}EI \\ \frac{\sqrt{3}}{4}\quad\frac{EA}{2}\quad\frac{3EI}{2} & \frac{5}{8}EA+\frac{9}{8}EI & \frac{3\sqrt{3}}{4}EI & \frac{3\sqrt{3}}{4}EI \\ \frac{3}{4}EI & \frac{3\sqrt{3}}{4}EI & 4EI & EI \\ -\frac{3}{4}EI & \frac{3\sqrt{3}}{4}EI & EI & 2EI \end{bmatrix} \begin{bmatrix} \mu_2 \\ v_2 \\ \theta_2 \\ \theta_3 \end{bmatrix} = \begin{bmatrix} \frac{3}{8}EA+\frac{3}{8}EI\ c \\ \frac{EA}{2}\quad\frac{3EI}{2}\quad\times\frac{\sqrt{3}}{4}c \\ 30KN\ \ m\quad\frac{3}{4}EIc \\ \frac{3}{4}EIc \end{bmatrix}$$

8-6 试采用先处理法列出图示刚架的结构刚度方程，并写出 CG 杆杆端力的矩阵表达式。设各杆的 EI=常数，忽略杆件的轴向变形。



解：（1）结构标识如上图。

单元	局部坐标系（i→ j）	杆长	cosα	sinα
①	2 → 3	5	4/5	3/5
②	3 → 5	6	1	0
③	6 → 7	6	0	-1
④	1 → 2	3	0	1
⑤	4 → 3	6	0	1

（2）建立结点位移向量，结点力向量

$$= \left[v_2 \quad \theta_2 \quad \theta_3 \quad \theta_5 \quad \theta_6 \right]$$

$$F = \left[15+10 \quad 0 \quad 0 \quad 0 \quad 0 \right]$$

（3）建立单元刚度矩阵（考虑杆件①及②两端点无相对水平位移，故水平位移可以不考虑）

$$k① = \begin{array}{cc} \theta_2 & \theta_3 \\ \frac{4EI}{l} & \frac{2EI}{l} \\ \frac{2EI}{l} & \frac{4EI}{l} \end{array}$$

其中 l=5m

$$k② = \begin{array}{cc} \theta_3 & \theta_5 \\ \frac{4EI}{l} & \frac{2EI}{l} \\ \frac{2EI}{l} & \frac{4EI}{l} \end{array}$$ 其中 l=6m

$$k③ = \begin{array}{cc} v_2 & \theta_6 \\ \frac{12EI}{l^3} & \frac{6EI}{l^2} \\ \frac{6EI}{l^2} & \frac{4EI}{l} \end{array}$$ 其中 l=6m

$$k④ = \begin{array}{cc} v_2 & \theta_2 \\ \frac{12EI}{l^3} & \frac{6EI}{l^2} \\ \frac{6EI}{l^2} & \frac{4EI}{l} \end{array}$$ 其中 l=3m

$$k⑤ = \begin{array}{cc} v_2 & \theta_3 \\ \frac{12EI}{l^3} & \frac{6EI}{l^2} \\ \frac{6EI}{l^2} & \frac{4EI}{l} \end{array}$$ 其中 l=6m

（4）建立结构刚度方程（按对号入座的方式）

$$\begin{array}{ccccc} v_2 & \theta_2 & \theta_3 & \theta_5 & \theta_6 \\ \frac{5}{9}EI & \frac{2}{3}EI & \frac{1}{6}EI & 0 & \frac{1}{6}EI \\ \frac{2}{3}EI & \frac{32}{15}EI & \frac{2}{5}EI & 0 & 0 \\ \frac{1}{6}EI & \frac{2}{5}EI & \frac{32}{15}EI & \frac{1}{3}EI & 0 \\ 0 & 0 & \frac{1}{3}EI & \frac{2}{3}EI & 0 \\ \frac{1}{6}EI & 0 & 0 & 0 & \frac{2}{3}EI \end{array} \begin{array}{c} v_2 \\ \theta_2 \\ \theta_3 \\ \theta_5 \\ \theta_6 \end{array} = \begin{array}{c} 25 \\ 0 \\ 0 \\ 0 \\ 0 \end{array}$$

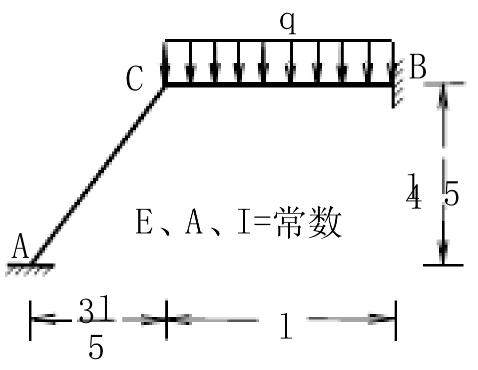
(方程中已省去单位)

解得：
$$\begin{array}{c} v_2 \\ \theta_2 \\ \theta_3 \\ \theta_5 \\ \theta_6 \end{array} = \begin{array}{c} 82.06 \\ 25.31 \\ 1.81 \\ 0.90 \\ 20.52 \end{array} \frac{1}{EI}$$

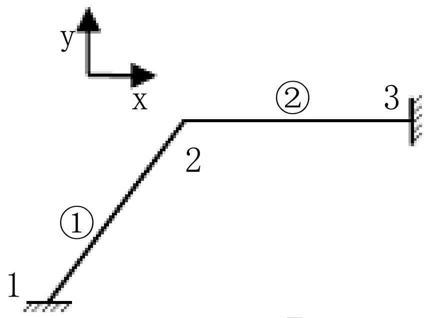
（5）写出 CG 杆杆端力的矩阵表达式

$$\overline{F}^{③} = \begin{Bmatrix} \mu_6 \\ v_6 \\ \theta_6 \\ \mu_7 \\ v_7 \\ \theta_7 \end{Bmatrix} = \frac{EI}{6}\begin{bmatrix} \frac{12}{36} & 0 & \frac{6}{6} & -\frac{12}{36} & -0 & \frac{6}{6} - \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{6}{6} & 0 & 4 & \frac{6}{6} & 0 & 2 \\ -\frac{12}{36} & 0 & \frac{6}{6} & \frac{12}{36} & 0 & \frac{6}{6} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{6}{6} & 0 & 2 & \frac{6}{6} & 0 & 4 \end{bmatrix} \frac{1}{EI}\begin{Bmatrix} 82.06 \\ 0 \\ 20.52 \\ 0 \\ 0 \\ 0 \end{Bmatrix} = \begin{bmatrix} \frac{1}{18} & 0 & \frac{1}{6} & -\frac{1}{18} & -0 & \frac{1}{6} - \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{6} & 0 & \frac{2}{3} & \frac{1}{6} & 0 & \frac{1}{3} \\ -\frac{1}{18} & 0 & \frac{1}{6} & \frac{1}{18} & 0 & \frac{1}{6} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{6} & 0 & \frac{1}{3} & \frac{1}{6} & 0 & \frac{2}{3} \end{bmatrix} \begin{Bmatrix} 82.06 \\ 0 \\ 20.52 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

8-7 试采用矩阵位移法分析图示刚架，并作出刚架的内力图。设各杆件 E、A、I 相同，A=1000 I/l2。



解：（1）结构标识



$$A = \frac{1000I}{l^2}$$

单元	局部坐标系（$i \rightarrow j$）	杆长	cosα	sinα
①	$1 \rightarrow 2$	l	3/5	4/5
②	$2 \rightarrow 3$	l	1	0

（2）建立结点位移向量，结点力向量

$$= \begin{bmatrix} \mu_2 & v_2 & \theta_2 \end{bmatrix}^T$$

$$F = \begin{bmatrix} 0 & \frac{ql}{2} & \frac{ql^2}{12} \end{bmatrix}^T$$

（3）建立单元刚度矩阵

$$k^{①} = \begin{array}{c} \begin{array}{ccc} \mu_2 & v_2 & \theta_2 \end{array} \\ \begin{bmatrix} \frac{EA}{l}\times\left(\frac{3}{5}\right)^2+\frac{12EI}{l^3}\times\left(\frac{4}{5}\right)^2 & \frac{EA}{l} \quad \frac{12EI}{l^3} \quad \frac{12}{25} & \frac{6EI}{l^2}\times\frac{4}{5} \\ \frac{EA}{l} \quad \frac{12EI}{l^3} \quad \frac{12}{25} & \frac{EA}{l}\times\left(\frac{4}{5}\right)^2+\frac{12EI}{l^3}\times\left(\frac{3}{5}\right)^2 & \frac{6EI}{l^2}\times\frac{3}{5} \\ \frac{6EI}{l^2}\times\frac{4}{5} & \frac{6EI}{l^2}\times\frac{3}{5} & \frac{4EI}{l} \end{bmatrix} \end{array} = EI \begin{array}{c} \begin{array}{ccc} \mu_2 & v_2 & \theta_2 \end{array} \\ \begin{bmatrix} \frac{9192}{25l^3} & \frac{11856}{25l^3} & \frac{24}{5l^2} \\ \frac{11856}{25l^3} & \frac{16108}{25l^3} & \frac{18}{5l^2} \\ -\frac{24}{5l^2} & \frac{18}{5l^2} & \frac{4}{l} \end{bmatrix} \end{array}$$

$$k^{②}=\begin{array}{c} \mu_2 & v_2 & \theta_2 \\ \frac{1000EI}{25l^3} & 0 & 0 \\ 0 & \frac{12EI}{l^3} & \frac{6EI}{l^2} \\ 0 & \frac{6EI}{l^2} & \frac{4EI}{l} \end{array}$$

(4)建立结构刚度矩阵

$$k=EI\begin{bmatrix} \frac{34192}{25l^3} & \frac{11856}{25l^3} & -\frac{24}{5l^2} \\ \frac{11856}{25l^3} & \frac{16408}{25l^3} & -\frac{18}{5l^2} \\ -\frac{24}{5l^2} & \frac{12}{5l^2} & \frac{8}{l} \end{bmatrix}$$

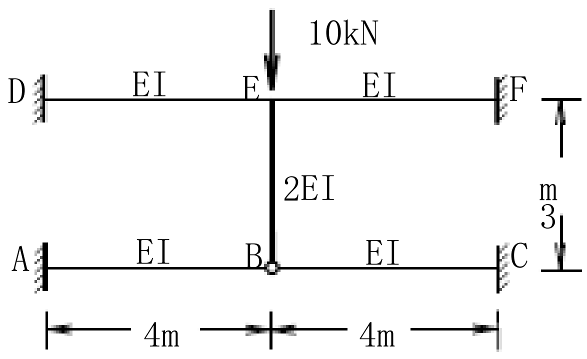
(5)结构刚度方程

$$k\begin{bmatrix}\mu_2\\ v_2\\ \theta_2\end{bmatrix}=\begin{bmatrix}0\\ \frac{ql}{2}\\ \frac{ql^2}{12}\end{bmatrix}$$

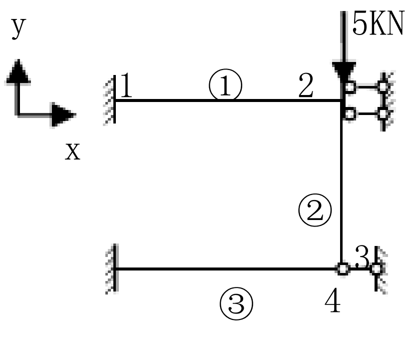
解得：$$\begin{bmatrix}\mu_2\\ v_2\\ \theta_2\end{bmatrix}=\begin{bmatrix}0.0003\\ 0.0009\\ 0.0100\end{bmatrix}$$

8-8 试利用对称性用先处理法分析图示刚架并作出 M 、F_Q 图。忽略杆件的轴向变形。

(a)



解：(1）结构标识（取半结构）



单元	局部坐标系（$i \to j$）	杆长	$\cos\alpha$	$\sin\alpha$
①	$1 \to 2$	4	1	0
②	$2 \to 3$	3	0	-1
③	$5 \to 4$	4	1	0

$$= \begin{bmatrix} v_2 & \theta_4 \end{bmatrix}$$

$$F = \begin{bmatrix} 5KN & 0 \end{bmatrix}$$

(2)建立单元刚度矩阵

$$k = \begin{array}{c} v_2 \\ \frac{12EI}{l^3} \end{array} \qquad l=4m$$

$$k_{③} = \begin{array}{cc} v_2 & \theta_4 \\ \frac{12EI}{l^3} & -\frac{6EI}{l^2} \\ -\frac{6EI}{l^2} & \frac{4EI}{l} \end{array} \qquad l=4m$$

(3)建立结构刚度矩阵

$$k = \begin{array}{cc} v_2 & \theta_4 \\ \frac{24EI}{l^3} & -\frac{6EI}{l^2} \\ -\frac{6EI}{l^2} & \frac{4EI}{l} \end{array}$$

(4)建立结构刚度方程

$$\begin{matrix} \frac{24EI}{l^3} & -\frac{6EI}{l^2} \\ -\frac{6EI}{l^2} & \frac{4EI}{l} \end{matrix} \quad \begin{matrix} v_2 \\ \theta_4 \end{matrix} = \begin{matrix} 5 \\ 0 \end{matrix}$$

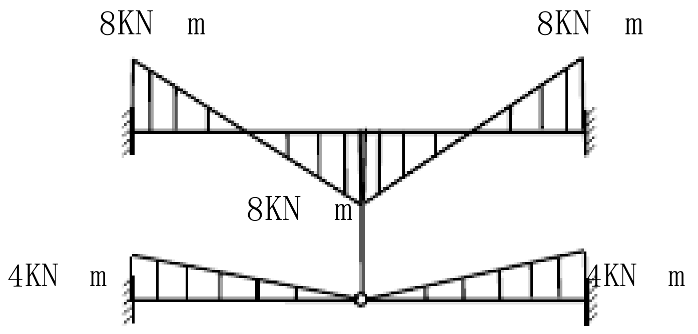
解得：$\begin{matrix} v_2 \\ \theta_4 \end{matrix} = \begin{matrix} -\frac{64}{3} \\ 8 \end{matrix} \ \frac{1}{EI}$

(5)计算杆件内力

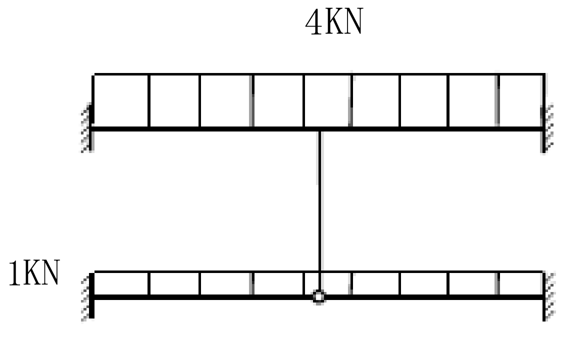
$$\overline{F}^{③} = F^{③} = \frac{EI}{l^2} \begin{matrix} \frac{12}{1} & 6 & \frac{12}{1} & 6 \\ 6 & 41 & 6 & -21 \\ -\frac{12}{1} & 6- & \frac{12}{1} & 6- \\ 6 & 21 & 6 & -41 \end{matrix} \begin{matrix} 0 \\ 0 \\ -\frac{64}{3} \\ -8 \end{matrix} \frac{1}{EI} = \begin{matrix} 1 \\ 4 \\ -1 \\ 0 \end{matrix} = \begin{matrix} Q_5 \\ M_5 \\ Q_4 \\ M_4 \end{matrix}$$

$$\overline{F}^{\{\}} = F^{\{\}} = \frac{EI}{l^2}\begin{bmatrix} \frac{12}{1} & 6 & \frac{12}{1} & 6 \\ 6 & 41 & 6 & -21 \\ -\frac{12}{1} & 6- & \frac{12}{1} & 6- \\ 6 & 21 & 6 & -41 \end{bmatrix}\begin{bmatrix}0\\0\\-\frac{64}{3}\\0\end{bmatrix}\frac{1}{EI} = \begin{bmatrix}4\\8\\-4\\8\end{bmatrix} = \begin{bmatrix}Q_1\\M_1\\Q_2\\M_2\end{bmatrix}$$

(6)作出 M 、F_Q 图

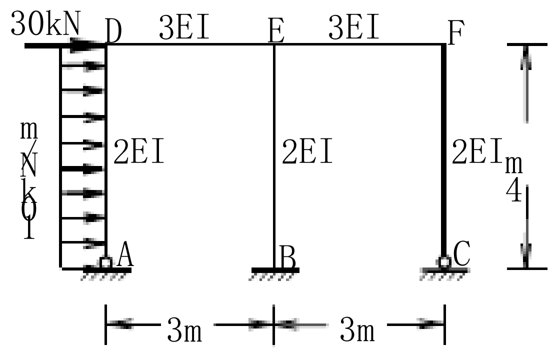


M

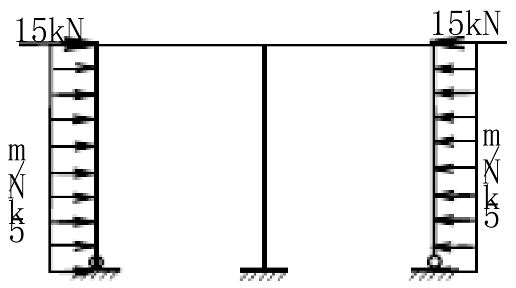


N

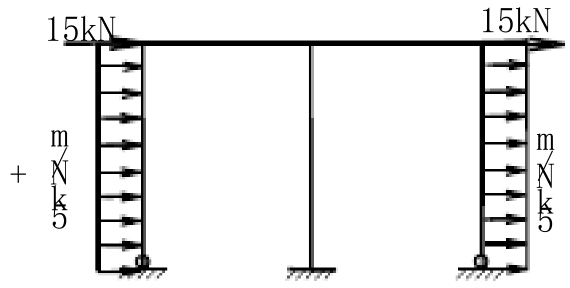
(b)



解：原结构等效为下面结构：



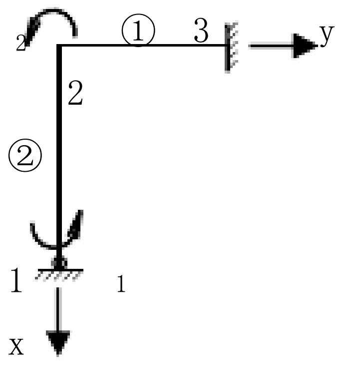
正对称



反对称

1.正对称结构

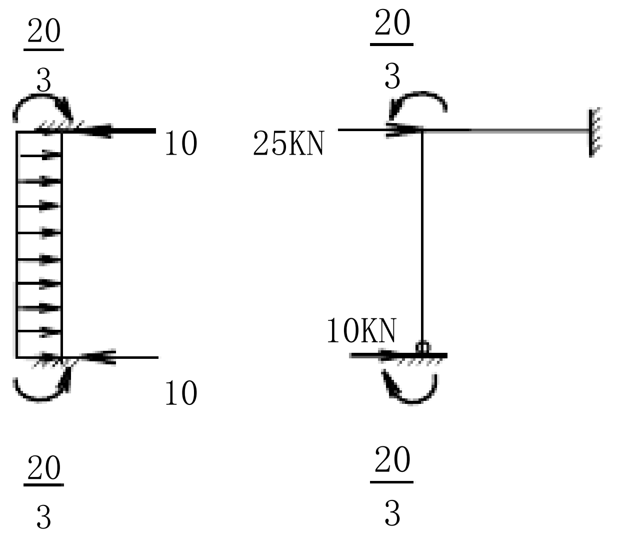
(1)结构标识如图所示



(2)结构位移向量

$$= \begin{bmatrix}\theta_1\\\theta_2\end{bmatrix} = \begin{bmatrix}{}_1\\{}_2\end{bmatrix}$$

⑶等效结点荷载



$$F = \frac{20}{3},\frac{20}{3}\ ^{T}$$

⑷建立单元刚度矩阵

$$k① = 2\times \begin{array}{cc} (\theta_2)_2 & (\theta_1)_1 \\ \frac{4EI}{1} & \frac{2EI}{1} \\ \frac{2EI}{1} & \frac{4EI}{1} \end{array} = 2EI \begin{array}{cc} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{array}$$

$$k② = 3\times \begin{array}{c} (\theta_2)_2 \\ \frac{4EI}{1} \end{array} = 3EI\ \frac{4}{3}$$

⑸建立结构刚度方程

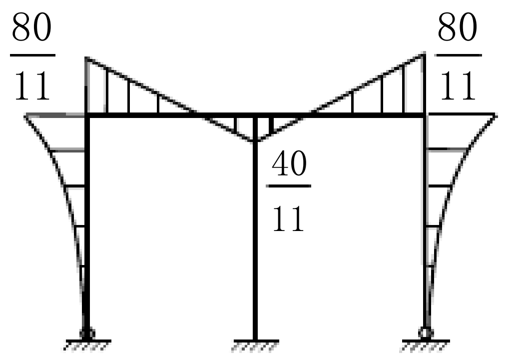
$$EI \begin{array}{cc} 2 & 1 \\ 1 & 6 \end{array} \begin{array}{c} {}_1 \\ {}_2 \end{array} = \begin{array}{c} \frac{20}{3} \\ \frac{20}{3} \end{array}$$

解得：${}_1 = \frac{140}{33EI}$， ${}_2 = \frac{20}{11EI}$

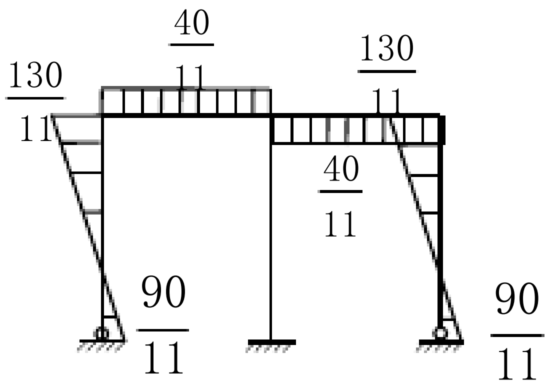
⑹求杆端力

$$\overline{F} = \begin{array}{c} \overline{F}_{y2} \\ \overline{M}_2 \\ \overline{F}_{y1} \\ \overline{M}_1 \end{array} = \begin{array}{c} -10 \\ -\frac{20}{3} \\ -10 \\ \frac{20}{3} \end{array} + 2EI \begin{array}{cccc} \frac{3}{16} & \frac{3}{8} & -\frac{3}{16} & \frac{3}{8} \\ \frac{3}{8} & 1 & -\frac{3}{8} & \frac{1}{2} \\ -\frac{3}{16} & -\frac{3}{8} & \frac{3}{16} & -\frac{3}{8} \\ \frac{3}{8} & \frac{1}{2} & -\frac{3}{8} & 1 \end{array} \begin{array}{c} 0 \\ {}_2 \\ 0 \\ {}_1 \end{array} = \begin{array}{c} -\frac{130}{11} \\ -\frac{80}{11} \\ -\frac{90}{11} \\ 0 \end{array}$$

$$\overline{F} = 3EI\begin{bmatrix}\frac{4}{9} & \frac{2}{3} & -\frac{4}{9} & \frac{2}{3}\\ \frac{2}{3} & \frac{4}{3} & -\frac{3}{8} & \frac{2}{3}\\ -\frac{4}{9} & -\frac{2}{3} & \frac{4}{9} & -\frac{2}{3}\\ \frac{2}{3} & \frac{2}{3} & -\frac{2}{3} & \frac{4}{3}\end{bmatrix}\begin{matrix}0\\ {}^{2}\\ 0\\ {}^{1}\end{matrix} = \begin{bmatrix}\frac{40}{11}\\ \frac{80}{11}\\ -\frac{40}{11}\\ \frac{40}{11}\end{bmatrix}$$



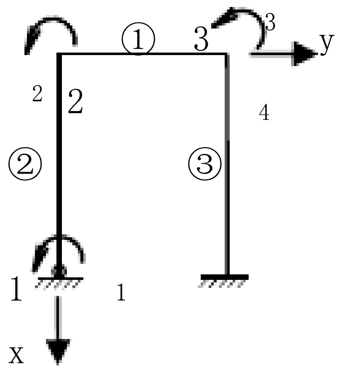
M 图



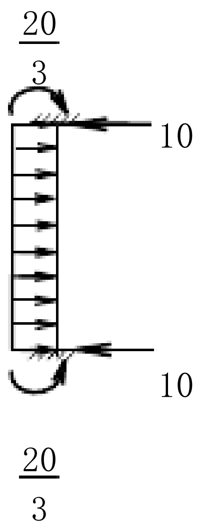
Q 图

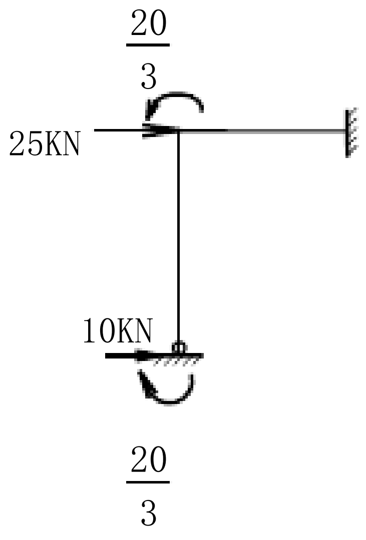
2. 反对称结构

(1)结构标识如图所示



(2)结构位移向量





$$=\begin{bmatrix}\theta_1\\ \theta_2\\ \theta_3\\ y_3\end{bmatrix}=\begin{bmatrix}{}_1\\ {}_2\\ {}_3\\ {}_4\end{bmatrix},F=\begin{bmatrix}\frac{20}{3}\\ \frac{20}{3}\\ 0\\ 25\end{bmatrix}=\begin{bmatrix}M_1\\ M_2\\ M_3\\ y_3\end{bmatrix}$$

(3)计算单元刚度矩阵

$$
k = EI \begin{array}{ccc}
4\ (y_3) & 2\ (\theta_2) & 1\ (\theta_1) \\
\frac{3}{8} & \frac{3}{4} & \frac{3}{4} \\
\frac{3}{4} & 2 & 1 \\
\frac{3}{4} & 1 & 2
\end{array}
$$

$$
k = EI \begin{array}{cc}
2\ (\theta_2) & 3\ (\theta_3) \\
4 & 2 \\
2 & 4
\end{array}
$$

$$
k = EI \begin{array}{cc}
4\ (y_3) & 3\ (\theta_3) \\
\frac{3}{16} & \frac{3}{8} \\
\frac{3}{8} & 1
\end{array}
$$

(4)建立刚度方程

$$
EI \begin{array}{cccc}
2 & 1 & 0 & \frac{3}{4} \\
1 & 6 & 2 & \frac{3}{4} \\
0 & 2 & 5 & \frac{3}{8} \\
\frac{3}{4} & \frac{3}{4} & \frac{3}{8} & \frac{9}{16}
\end{array}
\begin{array}{c}
{}_1 \\ {}_2 \\ {}_3 \\ {}_4
\end{array}
= \begin{array}{c}
\frac{20}{3} \\ \frac{20}{3} \\ 0 \\ 25
\end{array}
$$

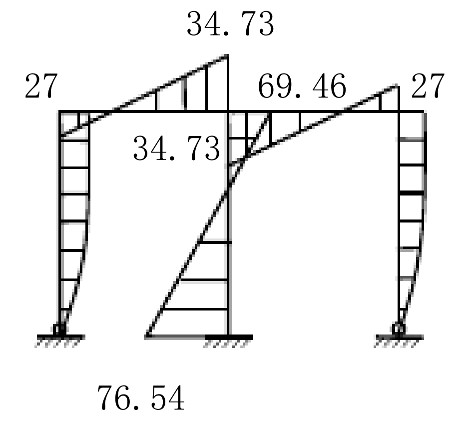
解得：${}_1 = 43.54\frac{1}{EI}$， ${}_2 = 3.21\frac{1}{EI}$， ${}_3 = 7.08\frac{1}{EI}$， ${}_4 = 111.50\frac{1}{EI}$

(5)求杆端力

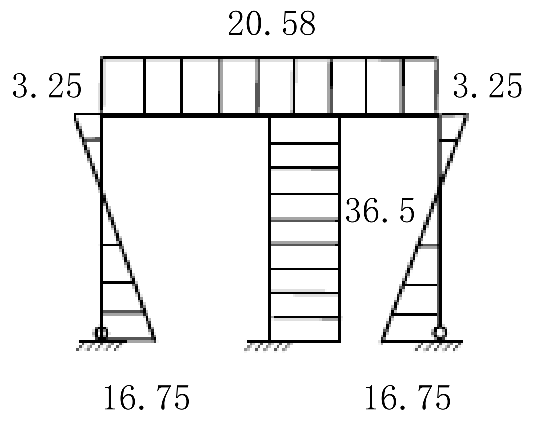
$$
\overline{F} = \begin{array}{c}
\overline{F}_{y2} \\ \overline{M}_2 \\ \overline{F}_{y1} \\ \overline{M}_1
\end{array}
= \begin{array}{c}
-10 \\ -\frac{20}{3} \\ -10 \\ \frac{20}{3}
\end{array}
+ 2EI \begin{array}{cccc}
\frac{3}{16} & \frac{3}{8} & -\frac{3}{16} & \frac{3}{8} \\
\frac{3}{8} & 1 & -\frac{3}{8} & \frac{1}{2} \\
-\frac{3}{16} & -\frac{3}{8} & \frac{3}{16} & -\frac{3}{8} \\
\frac{3}{8} & \frac{1}{2} & -\frac{3}{8} & 1
\end{array}
\begin{array}{c}
{}_4 \\ {}_2 \\ 0 \\ {}_1
\end{array}
= \begin{array}{c}
-3.25 \\ 27 \\ -16.75 \\ 0
\end{array}
$$

$$\overline{F} = \begin{bmatrix}\overline{F}_{y2}\\ \overline{M}_{2}\\ \overline{F}_{y3}\\ \overline{M}_{3}\end{bmatrix} = 3EI\begin{bmatrix}\frac{4}{9} & \frac{2}{3} & -\frac{4}{9} & \frac{2}{3}\\ \frac{2}{3} & \frac{4}{3} & -\frac{3}{8} & \frac{2}{3}\\ -\frac{4}{9} & -\frac{2}{3} & \frac{4}{9} & -\frac{2}{3}\\ \frac{2}{3} & \frac{2}{3} & -\frac{2}{3} & \frac{4}{3}\end{bmatrix}\begin{bmatrix}0\\ {}_{2}\\ 0\\ {}_{3}\end{bmatrix} = \begin{bmatrix}-20.58\\ -27\\ 20.58\\ -34.73\end{bmatrix}$$

$$\overline{F}^{③} = \begin{bmatrix}\overline{F}_{y3}\\ \overline{M}_{3}\\ \overline{F}_{y4}\\ \overline{M}_{4}\end{bmatrix} = EI\begin{bmatrix}\frac{3}{16} & \frac{3}{8} & -\frac{3}{16} & \frac{3}{8}\\ \frac{3}{8} & 1 & -\frac{3}{8} & \frac{1}{2}\\ -\frac{3}{16} & -\frac{3}{8} & \frac{3}{16} & -\frac{3}{8}\\ \frac{3}{8} & \frac{1}{2} & -\frac{3}{8} & 1\end{bmatrix}\begin{bmatrix}{}_{4}\\ {}_{3}\\ 0\\ 0\end{bmatrix} = \begin{bmatrix}18.25\\ 34.73\\ -18.25\\ 38.27\end{bmatrix}$$

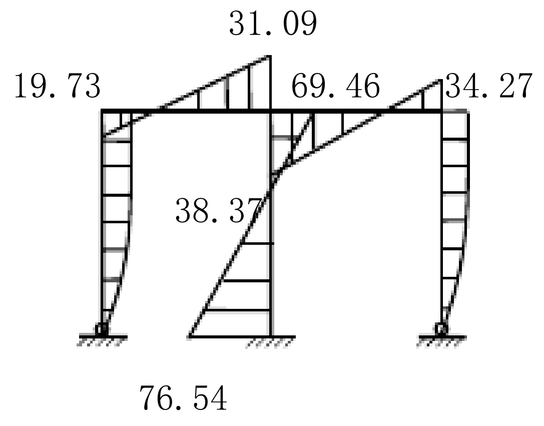


M 图

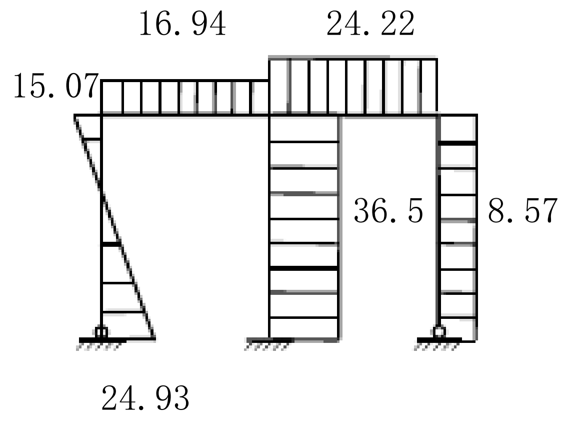


Q 图

整体受力图为：

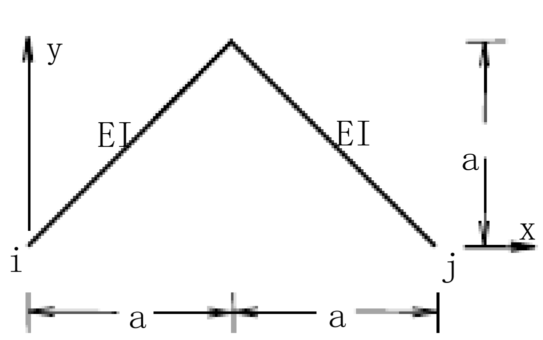


M 图

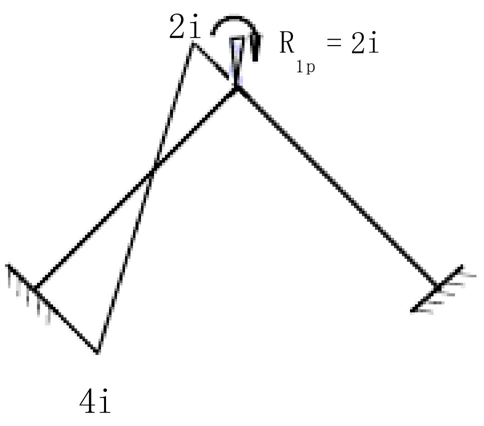


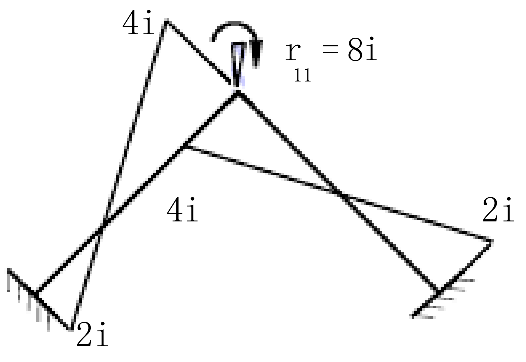
Q 图

8-9 设有如图两杆件刚结组成的特殊单元 i j(或称为子结构)，试直接根据单元刚度矩阵元素的物理意义，求出该特殊单元在图示坐标系中的刚度矩阵元素 k_{33} 和 k_{31}。



解：将单元在 3 方向转动单位角度视为主动力作用情况：（加一个刚臂）



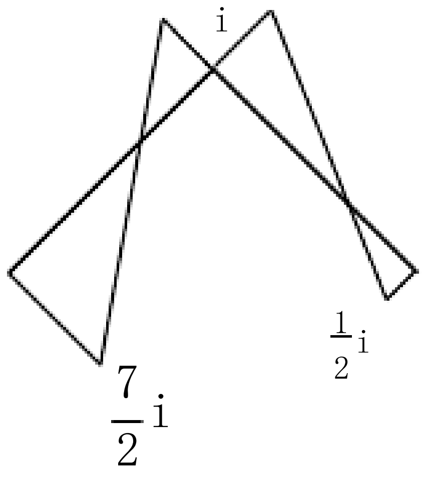


$$i=\frac{EI}{\sqrt{2}a}$$

M_p　　　　　　　　　　　　$\overline{M}_1$

$$8iZ_1+2i=0 \qquad Z_1=\frac{1}{4}$$

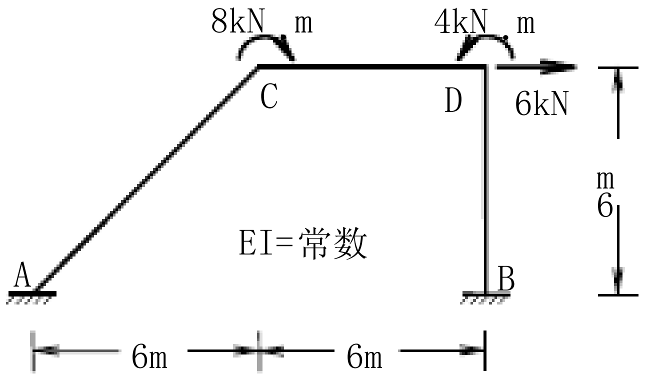
得出在 3 方向转动单位角度的弯矩图如下：



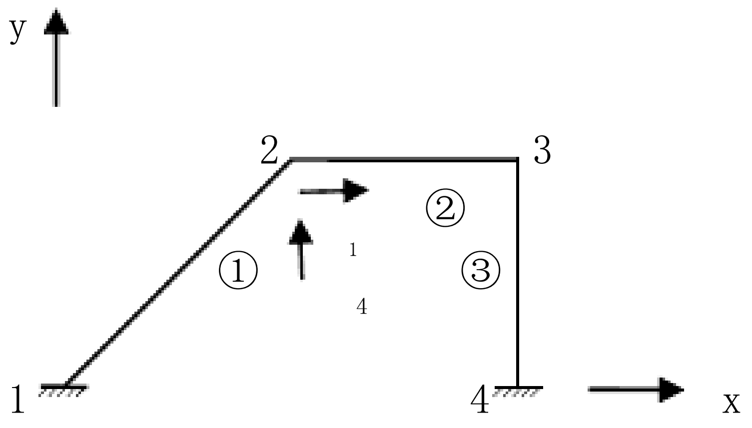
$$k_{33}=\frac{7}{2}i=\frac{7}{2}\frac{EI}{\sqrt{2}a}=\frac{7\sqrt{2}EI}{4a}$$

$$k_{31}=k_{13}=\frac{\frac{7}{2}+1\ /\sqrt{2}a}{\sqrt{2}}=\frac{9\sqrt{2}EI}{8a^2}$$

8-10 试采用先处理法列出图示刚架的结构刚度方程。设各杆的EI=常数，忽略杆件的轴向变形。



解：(1)结构标识如图



显而易见，$_4 = \quad _1$

(2)建立结构位移向量和结构荷载向量

$$= \left[\mu_2 \quad \theta_2 \quad \theta_3 \quad v_2\right]^T = \left[\quad _1 \quad _2 \quad _3 \quad _4 \right]^T$$

$$F = \left[6KN\ ,\ 8KN\ \ m, 4KN\ \ m, 0\right]^T$$

(3)建立单元刚度矩阵

$$k① = \begin{array}{ccc} 1 & 4 & 2 \\ \frac{\sqrt{2}}{24}EA + \frac{\sqrt{2}}{144}EI & \frac{\sqrt{2}}{24}EA \quad \frac{\sqrt{2}}{144}EI & \frac{\sqrt{2}}{24}EI \\ \frac{\sqrt{2}}{24}EA \quad \frac{\sqrt{2}}{144}EI & \frac{\sqrt{2}}{24}EA + \frac{\sqrt{2}}{144}EI & \frac{\sqrt{2}}{24}EI \\ \frac{\sqrt{2}}{24}EI & \frac{\sqrt{2}}{24}EI & \frac{\sqrt{2}}{3}EI \end{array}$$

$$k② = \begin{array}{ccc} 4 & 2 & 3 \\ \frac{EI}{18} & \frac{EI}{6} & \frac{EI}{6} \\ \frac{EI}{6} & \frac{2}{3}EI & \frac{EI}{3} \\ \frac{EI}{6} & \frac{EI}{3} & \frac{2}{3}EI \end{array}$$

$$k③ = \begin{array}{cc} 1 & 3 \\ \frac{EI}{18} & \frac{EI}{6} \\ \frac{EI}{6} & \frac{2}{3}EI \end{array}$$

(4)建立结构刚度方程

将上述单元刚度矩阵的元素，按照其对应的未知节点位移序号对号入座，即可得到结构刚度矩阵，据此可列出结构的刚度方程。

$$\begin{matrix} \frac{\sqrt{2}}{24}EA+\frac{8+\sqrt{2}}{144}EI & \frac{\sqrt{2}}{24}EI & \frac{1}{6}EI & \frac{\sqrt{2}}{24}EA \quad \frac{\sqrt{2}}{144}EI \\ \frac{\sqrt{2}}{24}EI & \frac{2+\sqrt{2}}{3}EI & \frac{1}{3}EI & \frac{4 \quad \sqrt{2}}{24}EI \\ \frac{1}{6}EI & \frac{1}{3}EI & \frac{4}{3}EI & \frac{1}{6}EI \\ \frac{\sqrt{2}}{24}EA \quad \frac{\sqrt{2}}{144}EI & \frac{4 \quad \sqrt{2}}{24}EI & \frac{1}{6}EI & \frac{\sqrt{2}}{24}EA+\frac{8+\sqrt{2}}{144}EI \end{matrix} \quad \begin{matrix} {}_1 \\ {}_2 \\ {}_3 \\ {}_4 \end{matrix} = \begin{matrix} 6KN \\ 8KN \quad m \\ 4KN \quad m \\ 0 \end{matrix}$$

将 $_4$ = $_1$ 带入上式，然后将结构刚度矩阵第一列减去第四列得方程。

$$EI \begin{matrix} \frac{8+\sqrt{2}}{144} & \frac{\sqrt{2}}{24} & \frac{1}{6} \\ \frac{\sqrt{2} \quad 2}{12} & \frac{2+\sqrt{2}}{3} & \frac{1}{3} \\ 0 & \frac{1}{3} & \frac{4}{3} \\ \frac{8+\sqrt{2}}{144} & \frac{4 \quad \sqrt{2}}{24} & \frac{1}{6} \end{matrix} \quad \begin{matrix} {}_1 \\ {}_2 \\ {}_3 \end{matrix} = \begin{matrix} 6KN \\ 8KN \quad m \\ 4KN \quad m \\ 0 \end{matrix}$$

上述方程组四个方程，三个未知数，为了获得位移解的存在性，以及刚度矩阵的对称性，我们将第一个方程减去第四个方程，得：

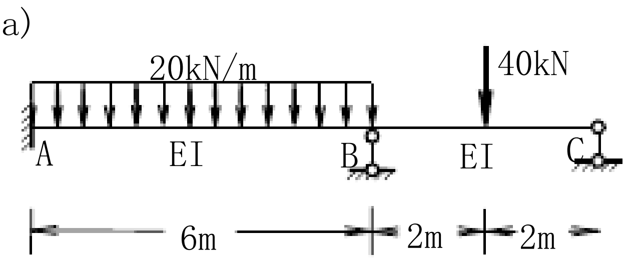
$$EI \begin{matrix} \frac{8+\sqrt{2}}{12} & \frac{\sqrt{2} \quad 2}{12} & 0 \\ \frac{\sqrt{2} \quad 2}{12} & \frac{2+\sqrt{2}}{3} & \frac{1}{3} \\ 0 & \frac{1}{3} & \frac{4}{3} \end{matrix} \quad \begin{matrix} \mu_1 \\ \theta_2 \\ \theta_3 \end{matrix} = \begin{matrix} 6KN \\ 8KN \quad m \\ 4KN \quad m \end{matrix}$$

同济大学朱慈勉 结构力学 第9章超静定结构的实用计算方法与概念分析习题答案

9-1 试说出何为杆端转动刚度、弯矩分配系数和传递系数，为什么弯矩分配法一般只能用于无结点线位移的梁和刚架计算。

9-2 试用弯矩分配法计算图示梁和刚架，作出M 图，并求刚结点B 的转角ϕ_B。

(a)



解：设EI=6，则$i_{AB}=1, i_{BC}=1.5$

$$\mu_{BA}=\frac{4\times1}{4\times1+3\times1.5}=0.47$$

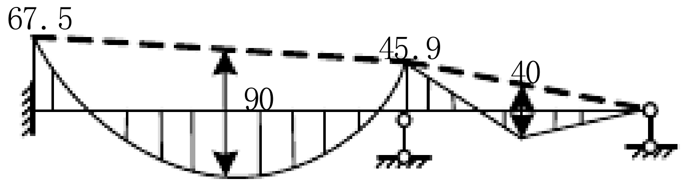
$$\mu_{BC}=\frac{3\times1.5}{4\times1+3\times1.5}=0.53$$

结点	A	B		C
杆端	AB	BA	BC	
分配系数	固端	0.47	0.53	绞支
固端弯矩	-60	60	-30	0
分配传递	-7.05 ←	-14.1	-15.9	0
最后弯矩	-67.05	45.9	-45.9	0

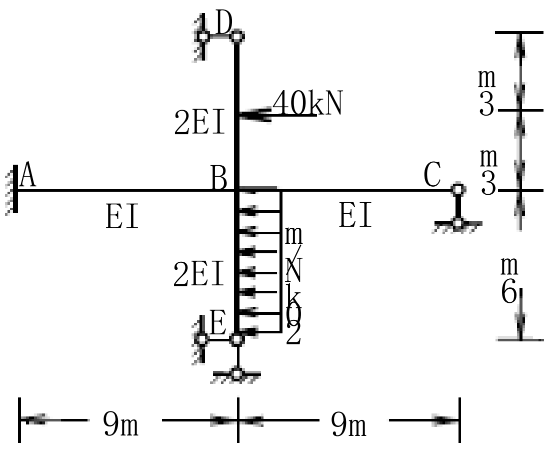
$$\theta_B=\frac{1}{3i}\quad M_{BA}\quad m_{BA}\quad \frac{1}{2}\left(M_{AB}\quad m_{AB}\right)$$

$$=\frac{2}{EI}\quad 45.9\quad 60\quad \frac{1}{2}\left(67.05+60\right)$$

$$=\quad\frac{21.15}{EI}KN\quad m_2\left(逆时针方向\right)$$



(b)



|

解：设EI=9，则