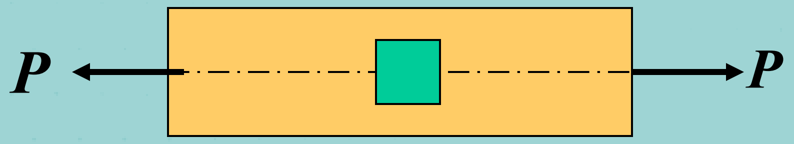
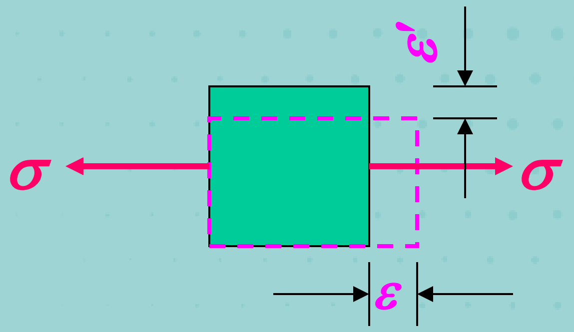
§7–8　广义胡克定律

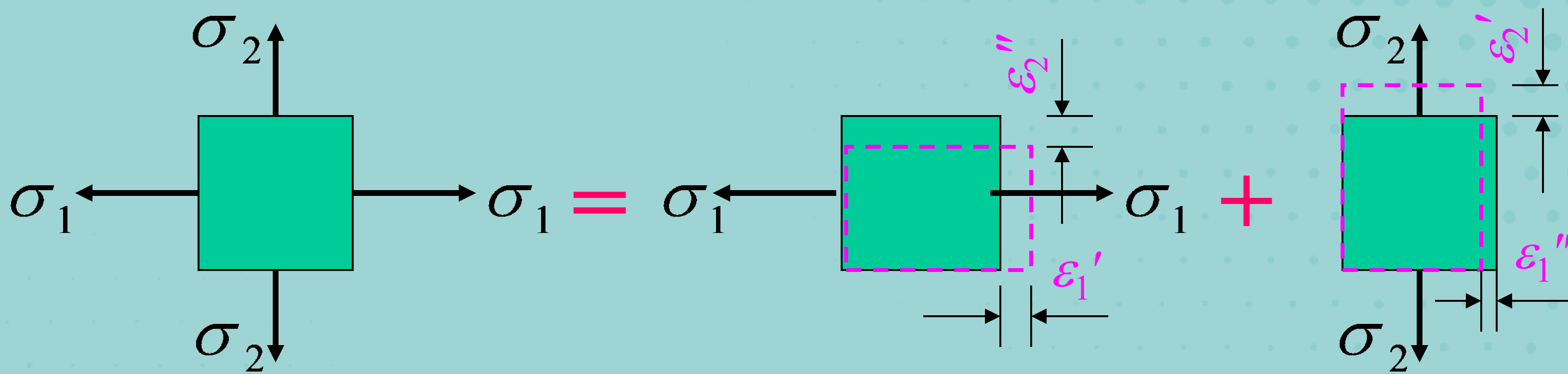




$$\varepsilon = \frac{\sigma}{E}$$

$$\varepsilon' = -\nu\varepsilon = -\frac{\nu}{E}\sigma$$

一、平面应力状态的广义胡克定律



$$\varepsilon_1=\varepsilon_1'+\varepsilon_1''=\frac{1}{E}(\sigma_1-\nu\sigma_2)$$

$$\varepsilon_1'=\frac{\sigma_1}{E}$$

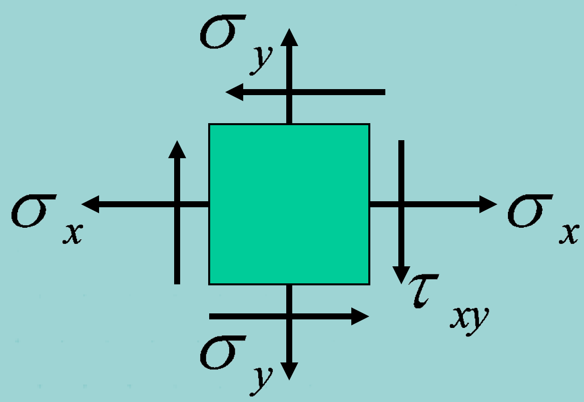
$$\varepsilon_1''=-\nu\varepsilon_2'=-\frac{\nu}{E}\sigma_2$$

$$\varepsilon_2=\varepsilon_2'+\varepsilon_2''=\frac{1}{E}(\sigma_2-\nu\sigma_1)$$

$$\varepsilon_2''=-\nu\varepsilon_1'=-\frac{\nu}{E}\sigma_1$$

$$\varepsilon_2'=\frac{\sigma_2}{E}$$

$$\left.\begin{aligned}\varepsilon_1&=\frac{1}{E}(\sigma_1-\nu\sigma_2)\\ \varepsilon_2&=\frac{1}{E}(\sigma_1-\nu\sigma_1)\end{aligned}\right\}$$

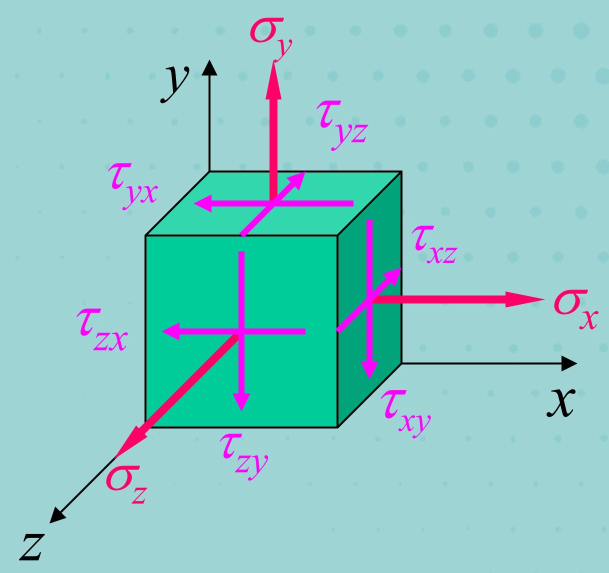


正应变只跟正应力有关，与剪应力无关；剪应力只跟剪应力有关，与正应力无关；

$$
\left.
\begin{aligned}
\varepsilon_x &= \frac{1}{E}(\sigma_x - \nu\sigma_y) \\
\varepsilon_y &= \frac{1}{E}(\sigma_y - \nu\sigma_x) \\
\gamma &= \frac{1}{G}\tau_x
\end{aligned}
\right\}
$$

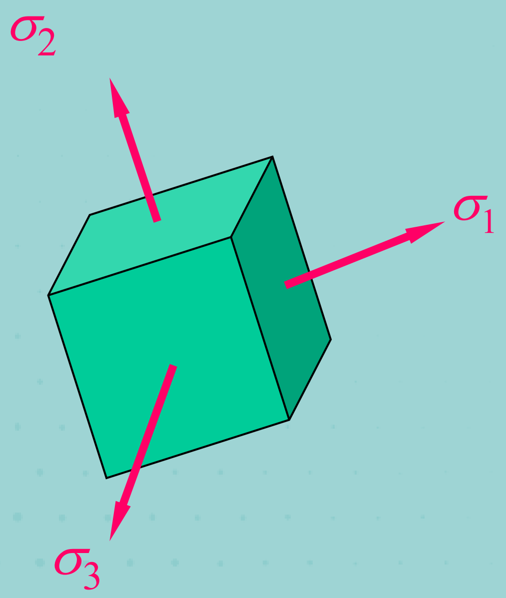
二、三向应力状态的广义胡克定律

$$\left.\begin{aligned}\varepsilon_x &= \frac{1}{E}[\sigma_x - \nu(\sigma_y + \sigma_z)]\\ \varepsilon_y &= \frac{1}{E}[\sigma_y - \nu(\sigma_z + \sigma_x)]\\ \varepsilon_z &= \frac{1}{E}[\sigma_z - \nu(\sigma_x + \sigma_y)]\end{aligned}\right\}$$



$$\gamma_{xy} = \frac{\tau_{xy}}{G} \qquad \gamma_{yz} = \frac{\tau_{yz}}{G} \qquad \gamma_{zx} = \frac{\tau_{zx}}{G}$$

三、主应力状态的广义胡克定律



$$
\left.
\begin{aligned}
\varepsilon_1 &= \frac{1}{E}[\sigma_1 - \nu(\sigma_2 + \sigma_3)] \\
\varepsilon_2 &= \frac{1}{E}[\sigma_2 - \nu(\sigma_3 + \sigma_1)] \\
\varepsilon_3 &= \frac{1}{E}[\sigma_3 - \nu(\sigma_1 + \sigma_2)]
\end{aligned}
\right\}
$$

四、应力--应变关系

$$\sigma_x = \frac{E}{1-\mu^2}\left[\varepsilon_x + \mu(\varepsilon_y + \varepsilon_z)\right]$$

$$\sigma_y = \frac{E}{1-\mu^2}\left[\varepsilon_y + \mu(\varepsilon_z + \varepsilon_x)\right]$$

$$\sigma_z = \frac{E}{1-\mu^2}\left[\varepsilon_z + \mu(\varepsilon_x + \varepsilon_y)\right]$$

$$\tau_{xy} = G\gamma_{xy} \qquad \tau_{yz} = G\gamma_{yz} \qquad \tau_{xz} = G\gamma_{xz}$$

[例1] 已知一受力构件自由表面上某点处的两主应变值为ε_1=240×10^{-6}，ε_3=–160×10^{-6}。材料的弹性模量E =210GPa，泊松比ν =0.3。求该点处的主应力值数，并求另一应变ε_2的数值和方向。

解：因主应力和主应变相对应，则由题意可得：

$$\sigma_2 = 0$$

即为平面应力状态，有

$$\varepsilon_1 = \frac{1}{E}\left(\sigma_1 - \nu\sigma_3\right)$$

$$\varepsilon_3 = \frac{1}{E}\left(\sigma_3 - \nu\sigma_1\right)$$

联立两式可解得：

$$\sigma_1=\frac{E}{1-\nu^2}\left(\varepsilon_1+\nu\varepsilon_3\right)=\frac{210\times10^9}{1-0.3^2}\left(240-0.3\times160\right)\times10^{-6}$$

$$=44.3\text{MPa}$$

$$\sigma_3=\frac{E}{1-\nu^2}\left(\varepsilon_3+\nu\varepsilon_1\right)=\frac{210\times10^9}{1-0.3^2}\left(-160+0.3\times240\right)\times10^{-6}$$

$$=-20.3\text{MPa}$$

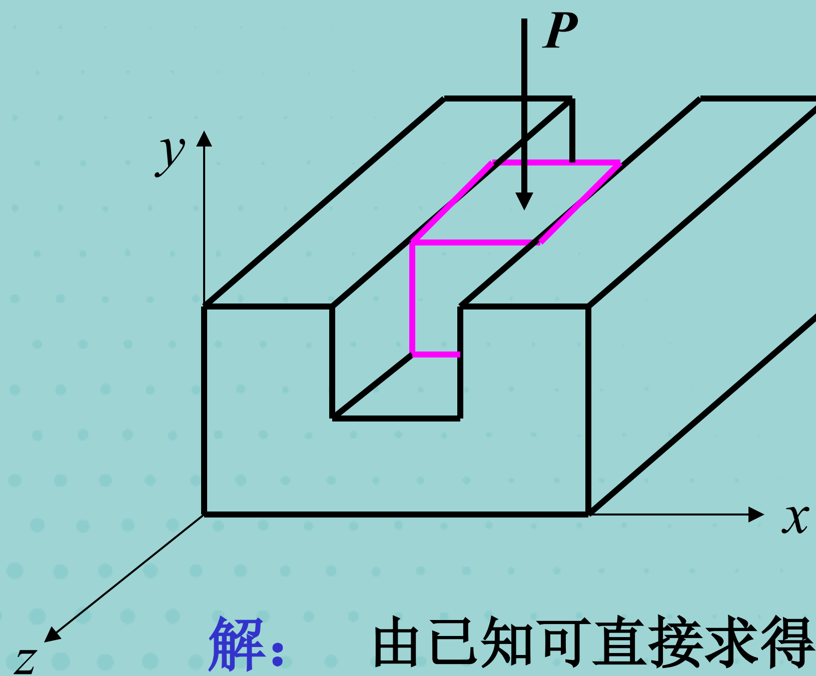
主应变ε_2为：

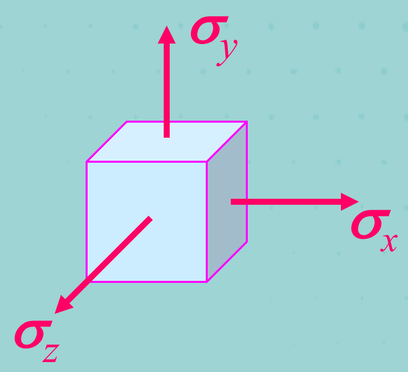
$$\varepsilon_2=-\frac{\nu}{E}\left(\sigma_1+\sigma_3\right)=-\frac{0.3}{210\times10^9}\left(44.3-20.3\right)\times10^6$$

$$=-34.3\times10^{-6}$$

其方向必与ε_1和ε_3垂直，沿构件表面的法线方向。

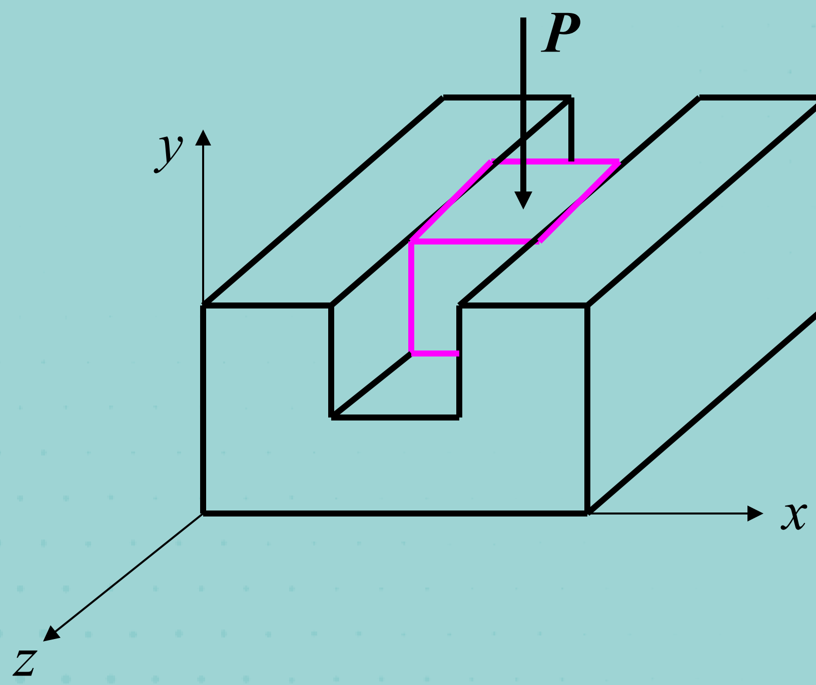
[例2]边长为a 的一立方钢块正好置于刚性槽中，钢块的弹性模量为E 、泊桑比为ν，顶面受铅直压力P 作用，求钢块的应力σ_x 、σ_y 、σ_z 和应变ε_x 、ε_y 、ε_z 。

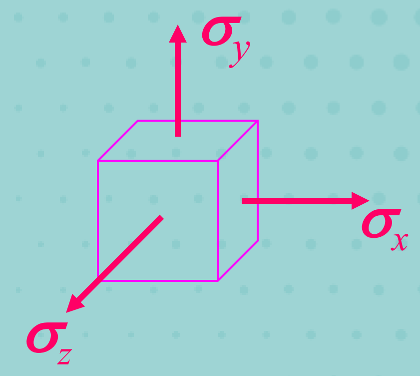




解： 由已知可直接求得：

$$\sigma_y = \frac{N}{A} = -\frac{P}{a^2}, \qquad \sigma_z = 0, \qquad \varepsilon_x = 0,$$





$$0=\frac{1}{E}[\sigma_x-\nu(\sigma_y+0)] \quad .$$

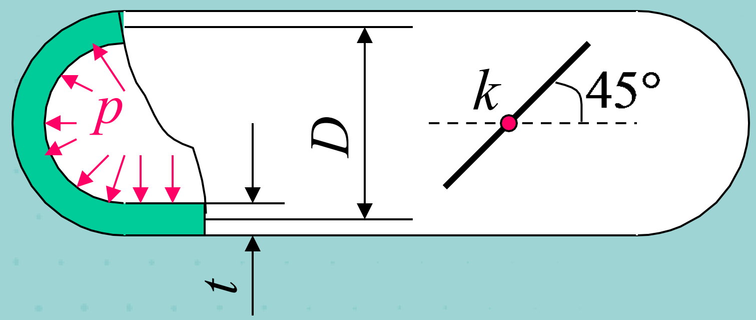
$$\varepsilon_y=\frac{1}{E}[\sigma_y-\nu(0+\sigma_x)]$$

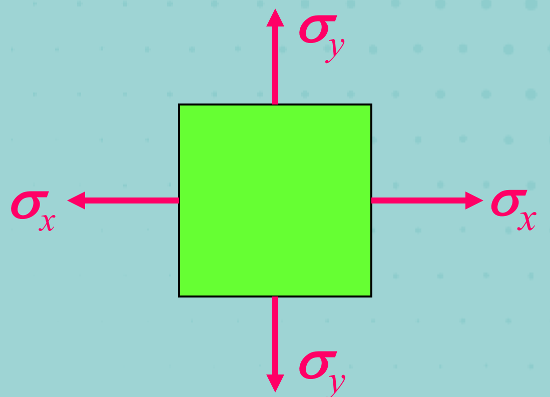
$$\varepsilon_z=\frac{1}{E}[0-\nu(\sigma_x+\sigma_y)] \quad .$$

$$\sigma_x=\nu\sigma_y=-\frac{\nu P}{a^2},$$

$$\varepsilon_y=\frac{1}{E}[\sigma_y-\nu^2\sigma_y]=-\frac{(1-\nu^2)P}{Ea^2},\quad \varepsilon_z=-\frac{\nu}{E}(\nu\sigma_y+\sigma_y)=\frac{\nu(1+\nu)P}{Ea^2}$$

[例3]薄壁筒内压容器($t/D\leq 1/20$)，筒的平均直径为D，壁厚为t，材料的E、ν已知。已测得筒壁上k点沿45°方向的线应变$\varepsilon_{45°}$，求筒内压强p。





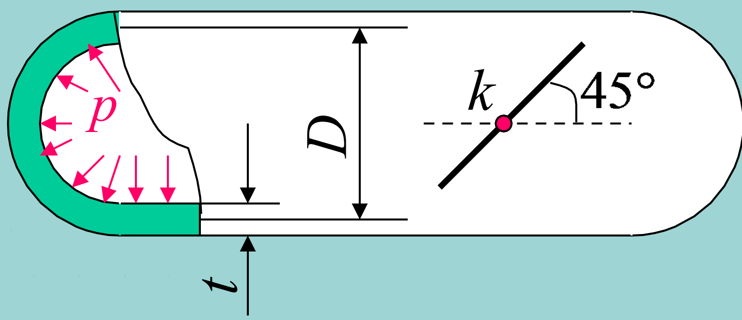
解:

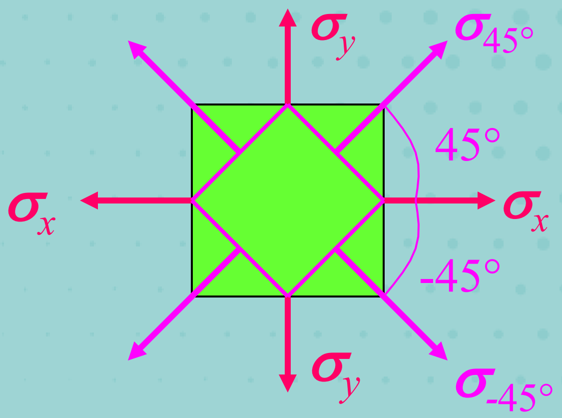
筒壁一点的轴向应力:

$$\sigma_x = \frac{p \cdot \dfrac{\pi D^2}{4}}{\pi D t} = \frac{pD}{4t}$$

筒壁一点的环向应力:

$$\sigma_y = \frac{pDl}{2tl} = \frac{pD}{2t}$$





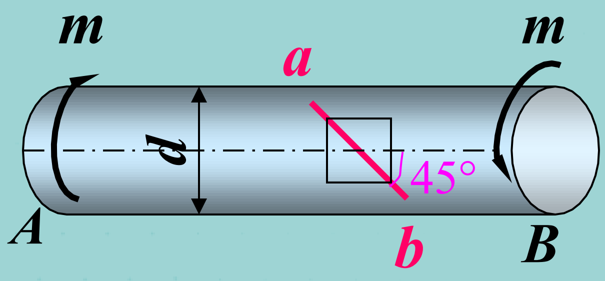
$$\sigma_x = \frac{pD}{4t}, \quad \sigma_y = \frac{pD}{2t}$$

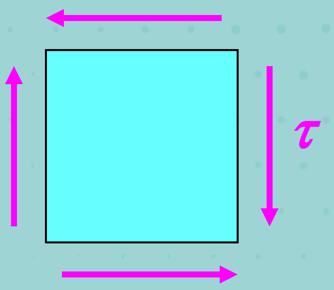
$$\sigma_{45°} = \sigma_{-45°} = \frac{\sigma_x + \sigma_y}{2} = \frac{3pD}{8t}$$

$$\varepsilon_{45°} = \frac{1}{E}(\sigma_{45°} - \nu\sigma_{-45°}) = \frac{1-\nu}{E} \cdot \frac{3pD}{8t}$$

$$\therefore p = \frac{8Et\varepsilon_{45°}}{3(1-\nu)D}$$

[例4]受扭圆轴如图所示，已知m 、 d 、 E、ν，求圆轴外表面沿ab方向的应变ε_{ab} 。

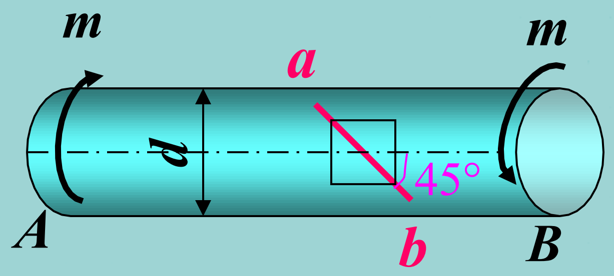


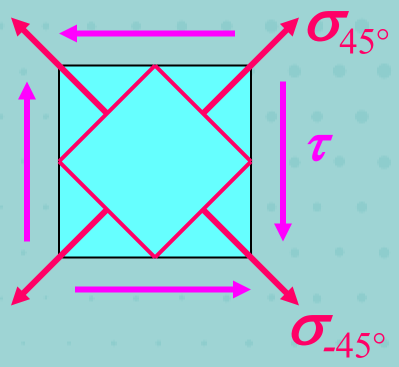


解:

$$\tau = \frac{T}{W_n} = \frac{16m}{\pi d^3}$$

$$\sigma_x = \sigma_y = 0, \quad \tau_x = \tau$$





$$\tau = \frac{T}{W_n} = \frac{16m}{\pi d^3}, \qquad \sigma_x = \sigma_y = 0, \quad \tau_x = \tau$$

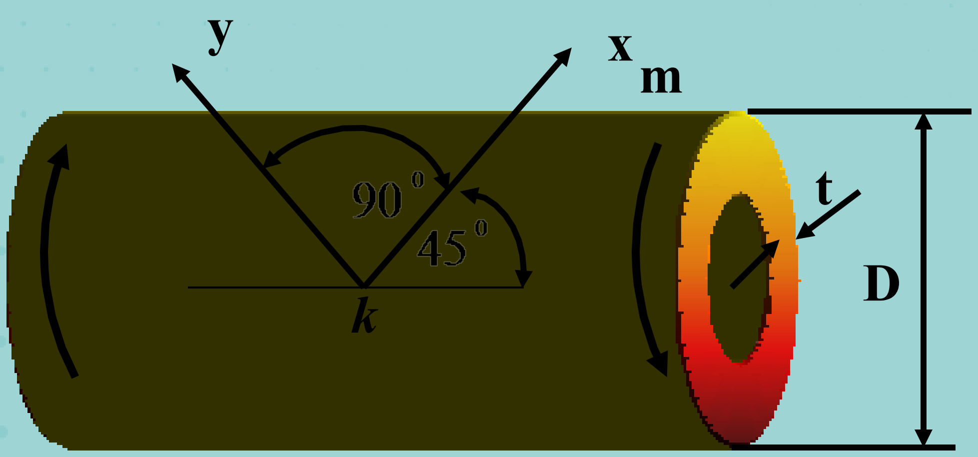
$$\sigma_{45^\circ} = -\tau_x \sin 90^\circ = -\tau$$

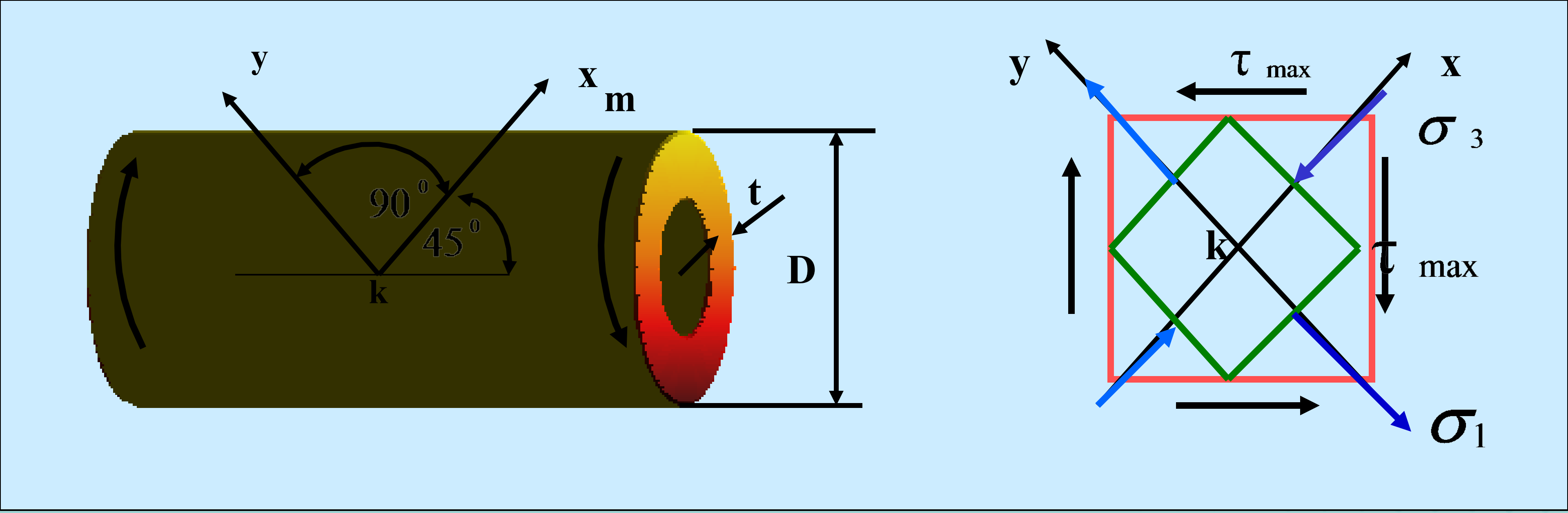
$$\sigma_{-45^\circ} = -\tau_x \sin(-90^\circ) = \tau$$

$$\varepsilon_{ab} = \frac{1}{E}(\sigma_{-45^\circ} - \nu\sigma_{45^\circ}) = \frac{1}{E}(\tau + \nu\tau)$$

$$= \frac{1+\nu}{E}\tau = \frac{16(1+\nu)m}{\pi E d^3}$$

[例5] 壁厚 t =10mm , 外径 D=60mm 的薄壁圆筒, 在表面上 k 点处与其轴线成 45°和135° 角即 x, y 两方向分别贴上应变片,然后在圆筒两端作用矩为 m 的扭转力偶,如图 所示已知圆筒材料的弹性模量为 $E = 200$GPa 和 $\nu = 0.3$,若该圆筒的变形在弹性范围内,且 $\tau_{max} =$ 80MPa，试求k点处的线应变 ε_x ,ε_y 以及变形后的筒壁厚度。





解：从圆筒表面 k 点处取出单元体, 如图 所示

可求得：

$$\sigma_y = \sigma_1 = \tau_{max} = 80\text{MPa}$$

$$\sigma_x = \sigma_3 = -\tau_{max} = -80\text{MPa}$$

$$\sigma_z = \sigma_2 = 0$$

k点处的线应变 ε_x，ε_y 为

$$\varepsilon_x = \frac{1}{E}(\sigma_x - \nu\sigma_y) = \frac{1}{E}(-\tau_{\max} - \nu\tau_{\max})$$

$$= -\frac{(1+\nu)}{E}\tau_{\max} = -5.2\times10^{-4}(\text{压应变})$$

$$\varepsilon_y = -\varepsilon_x = 5.2\times10^{-4}(\text{拉应变})$$

圆筒表面上k点处沿径向 (z轴) 的应变为

$$\varepsilon_z = -\frac{\nu}{E}(\sigma_x + \sigma_y) = -\frac{\nu}{E}(-\tau_{\max} + \tau_{\max}) = 0$$

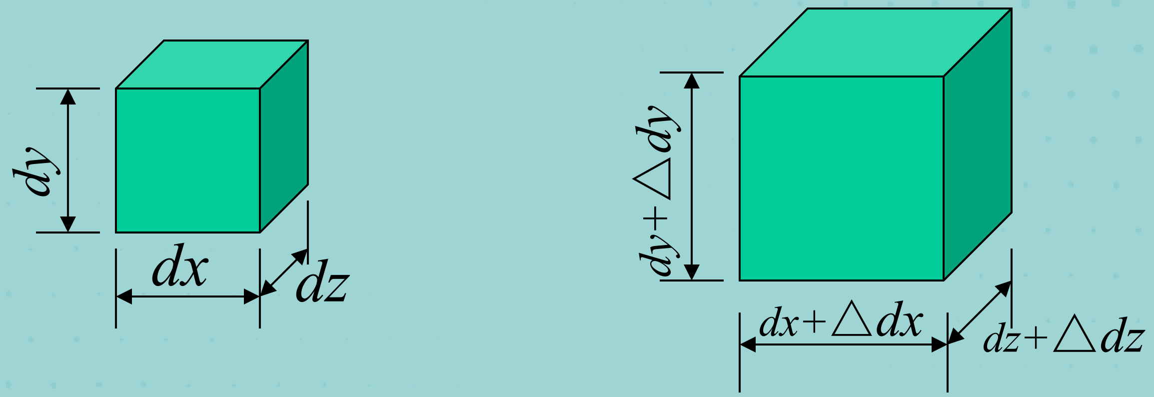
同理可得, 圆筒中任一点（该点到圆筒横截面中心的距离为ρ）处的径向应变为

$$\varepsilon_{z\rho} = -\frac{\nu}{E}(-\tau_\rho + \tau_\rho) = 0$$

★ 因此, 该圆筒变形后的厚度并无变化, 仍然为 t =10mm .

§7–9　复杂应力状态下的体积应变、比能

一、体积应变



$$V_0 = dxdydz$$

$$V_1 = (dx + \Delta dx)(dy + \Delta dy)(dz + \Delta dz)$$

$$= dxdydz(1 + \frac{\Delta dx}{dx})(1 + \frac{\Delta dy}{dy})(1 + \frac{\Delta dz}{dz})$$

$$= dxdydz(1 + \varepsilon_1)(1 + \varepsilon_2)(1 + \varepsilon_3)$$

$$V_0 = dxdydz$$

$$V_1 = dxdydz(1+\varepsilon_1)(1+\varepsilon_2)(1+\varepsilon_3)$$

略去高阶微量，得

$$V_1 = V_0(1+\varepsilon_1+\varepsilon_2+\varepsilon_3)$$

单元体的体积应变

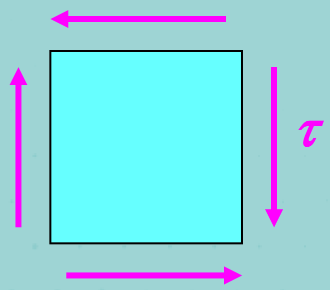
$$\varepsilon_V = \frac{V_1 - V_0}{V_0} = \varepsilon_1+\varepsilon_2+\varepsilon_3$$

代入式

$$\left.\begin{aligned} \varepsilon_1 &= \frac{1}{E}[\sigma_1 - \nu(\sigma_2+\sigma_3)] \\ \varepsilon_2 &= \frac{1}{E}[\sigma_2 - \nu(\sigma_3+\sigma_1)] \\ \varepsilon_3 &= \frac{1}{E}[\sigma_3 - \nu(\sigma_1+\sigma_2)] \end{aligned}\right\}$$

得：

$$\varepsilon_V = \frac{1-2\nu}{E}(\sigma_1+\sigma_2+\sigma_3)$$



纯剪应力状态： $\sigma_1=\tau,\quad \sigma_2=0,\quad \sigma_3=-\tau$

$$\varepsilon_V = 0$$

可见剪应力并不引起体积应变，对于非主应力单元体，其体积应变可改写为

$$\varepsilon_V = \frac{1-2\nu}{E}(\sigma_x+\sigma_y+\sigma_z)$$

体积应变只与三个主应力（正应力）之和有关，而与其比例无关。

令

$$\sigma_m = \frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3)$$

$$K = \frac{E}{3(1-2\nu)}$$

$$\boxed{\varepsilon_V = \frac{\sigma_m}{K}}$$

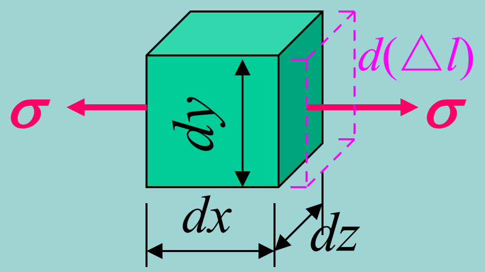
σ_m称为**平均正应力**，K 称为**体积弹性模量**。

二、比能

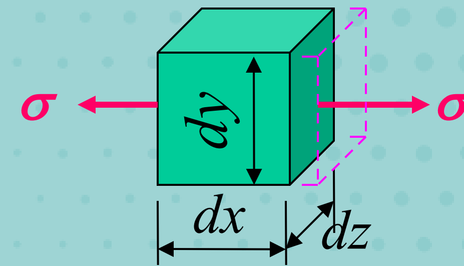
单位体积的变形能称为**变形能密度**，简称**比能**。

1. 单向拉压比能

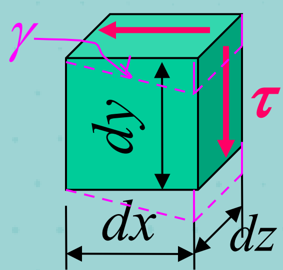
$$dU = \frac{1}{2}dN \cdot d(\Delta l) = \frac{1}{2}\sigma dydz \cdot \varepsilon dx$$



$$u=\frac{dU}{dV}=\frac{\frac{1}{2}\sigma dydz\cdot\varepsilon dx}{dxdydz}=\frac{1}{2}\sigma\varepsilon$$



2. 纯剪切比能



$$dU=\frac{1}{2}(\tau dydz)(\gamma dx)=\frac{1}{2}\tau\gamma(dx\sigma dydz)$$

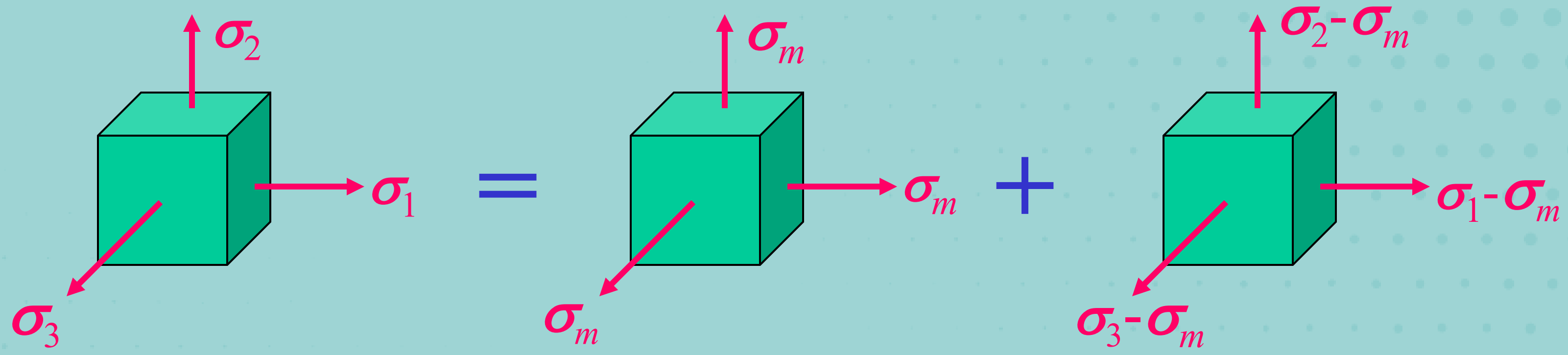
$$u=\frac{dU}{dV}=\frac{dU}{dxdydz}=\frac{1}{2}\tau\gamma$$

3. 复杂应力状态的比能

$$u=\frac{1}{2}(\sigma_1\varepsilon_1+\sigma_2\varepsilon_2+\sigma_3\varepsilon_3)$$

$$=\frac{1}{2E}[\sigma_1^2+\sigma_2^2+\sigma_3^2-2\nu(\sigma_1\sigma_2+\sigma_2\sigma_3+\sigma_3\sigma_1)]$$

4. 体积改变比能与形状改变比能



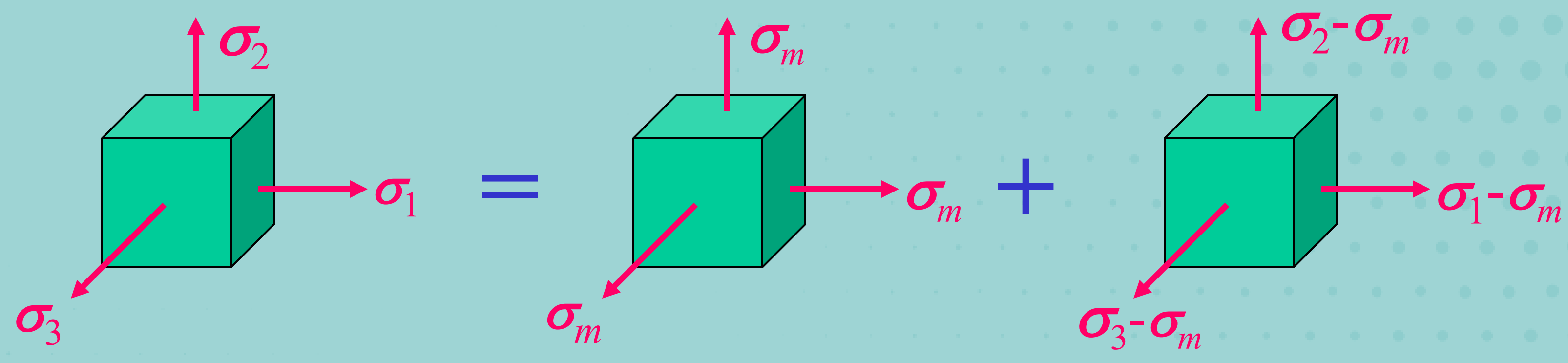
$u \quad = \quad u_V \quad + \quad u_f$

状态1受平均正应力σ_m作用，因各向均匀受力，故只有体积改变，而无形状改变，相应的比能称为**体积改变比能u_V**。

状态2的体积应变：

$$(\varepsilon_V)_2 = \frac{1-2\nu}{E}[(\sigma_1 - \sigma_m) + (\sigma_2 - \sigma_m) + (\sigma_3 - \sigma_m)] = 0$$

状态2无体积改变，只有形状改变，相应的比能称为**形状改变比能u_f**。



$$\boldsymbol{u} = \boldsymbol{u}_{\mathrm{V}} + \boldsymbol{u}_{\mathrm{f}}$$

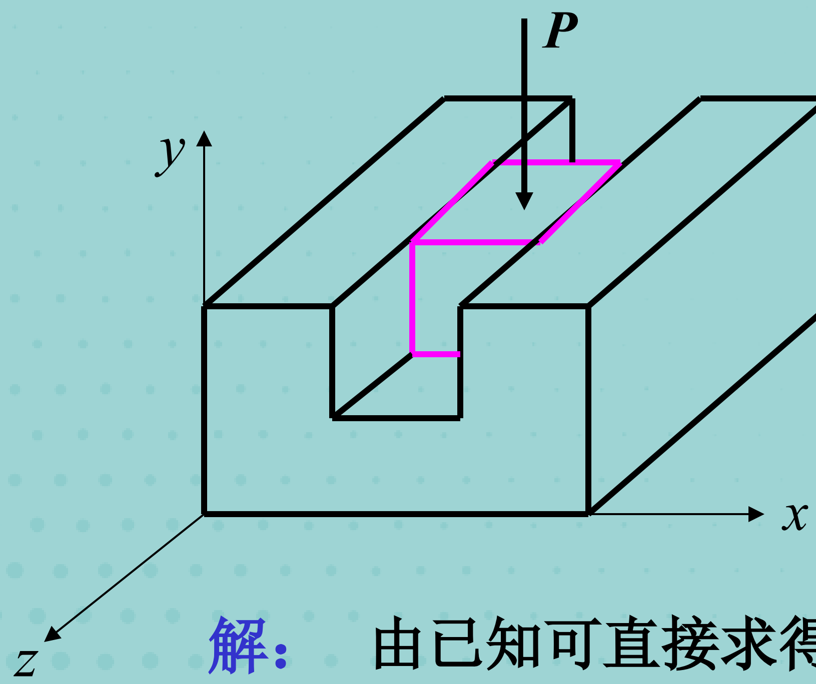
$$u_V = \frac{1}{2E}(3\sigma_m^2 - 2\nu \cdot 3\sigma_m^2) = \frac{1-2\nu}{2E}3\sigma_m^2 = \frac{1-2\nu}{6E}(\sigma_1 + \sigma_2 + \sigma_3)^2$$

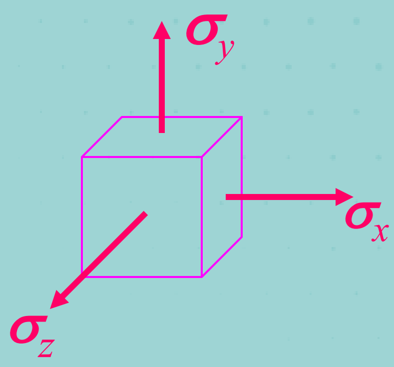
$$u_f = u - u_V = \frac{1}{2E}[\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\nu(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1)]$$

$$-\frac{1-2\nu}{6E}(\sigma_1 + \sigma_2 + \sigma_3)^2$$

$$\boxed{u_f = \frac{1+\nu}{6E}[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2]}$$

[例1]边长为a 的一立方钢块正好置于刚性槽中，钢块的弹性模量为E、泊桑比为ν，顶面受铅直压力P 作用，求钢块的体积应变ε_V和形状改变比能u_f。

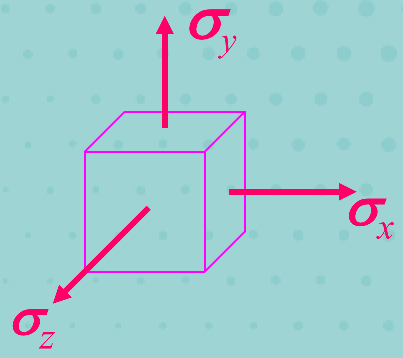




解： 由已知可直接求得：

$$\sigma_y = \frac{N}{A} = -\frac{P}{a^2}, \qquad \sigma_z = 0, \qquad \varepsilon_x = 0,$$

$$0=\frac{1}{E}[\sigma_x-\nu(\sigma_y+0)]$$



$$\sigma_x=\nu\sigma_y=-\frac{\nu P}{a^2},$$

$$\sigma_1=0,\quad \sigma_2=-\frac{\nu P}{a^2},\quad \sigma_3=-\frac{P}{a^2}$$

$$\varepsilon_V=\frac{1-2\nu}{E}(\sigma_1+\sigma_2+\sigma_3)=\frac{1-2\nu}{E}(0-\frac{\nu P}{a^2}-\frac{P}{a^2})$$

$$=-\frac{(1-2\nu)(1+\nu)P}{Ea^2}$$

$$u_f=\frac{1+\nu}{6E}[(\frac{\nu P}{a^2})^2+(-\frac{\nu P}{a^2}+\frac{P}{a^2})^2+(-\frac{P}{a^2})^2]$$

$$=\frac{(1+\nu)(1-\nu+\nu^2)P^2}{3Ea^4}$$