Chapter 6

Determinants

Overview

In this chapter we introduce idea of the determinant of a square matrix. We also investigate some of the properties of the determinant. For example, a square matrix is singular if and only if its determinant is zero.

Core sections

- Cofactor expansions of determinants
- Elementary operations and determinants
- Cramer's rule
- Applications of determinants: inverses and Wronskians

6.2 Cofactor expansions of determinants

If A is an **($n{\times}n$)** matrix, the determinant of A, denoted **det(A)** or $|A|$, is a number that we associate with A. determinants are usually defined either in terms of ***cofactors*** or in terms of ***permutations***.

Definition6.2.1: Let $A=(a_{ij})$ be a **($2{\times}2$)** matrix. The determinant of A is given by

$$\det(A) = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}.$$

The method of (2×2) determinants:

(1) $A = a_{11}a_{22} - a_{12}a_{21}$

$$\triangleq \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

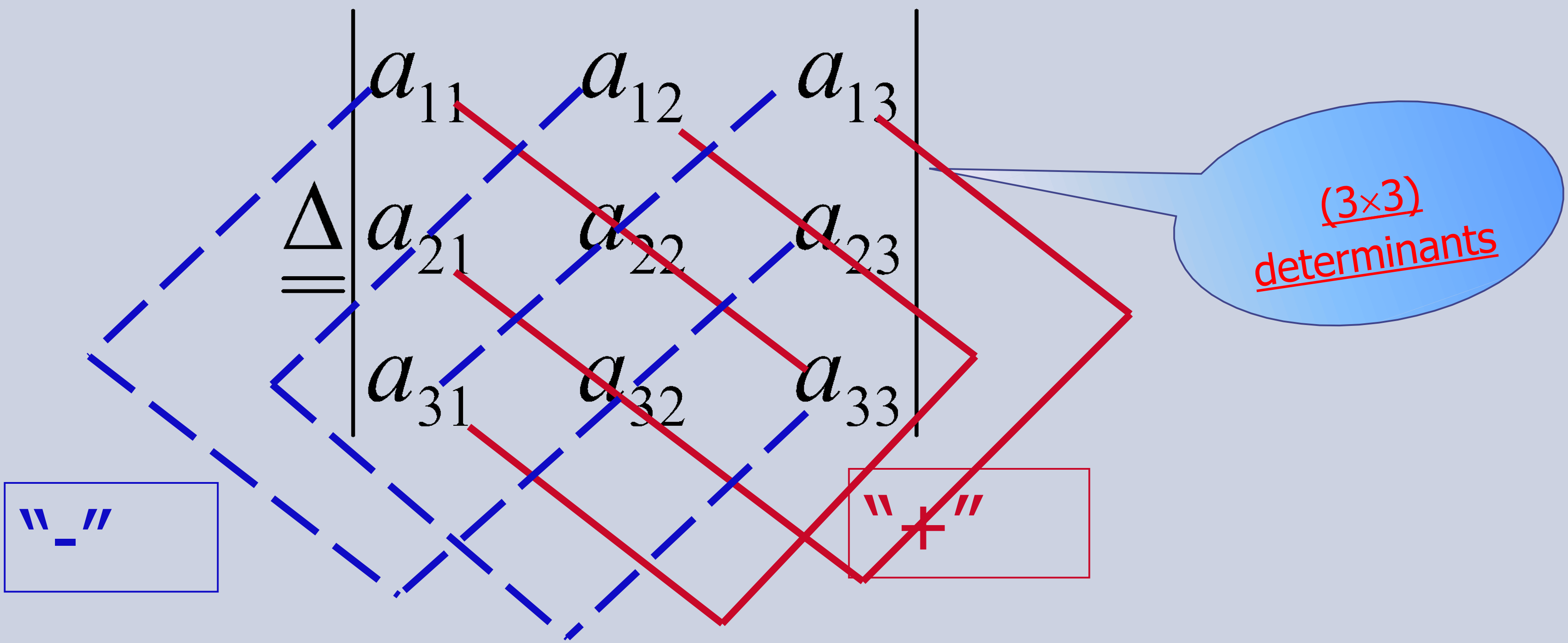
"-"

"+"

The method of (3×3) determinants:

$$A = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}$$

$$\triangleq \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$



Definition6.2.2: Let $A=(a_{ij})$ be an $(n \times n)$ matrix, and let M_{rs} denote the $[(n-1)\times(n-1)]$ matrix obtained by deleting the rth row and sth column form A. then M_{rs} is called a **minor matrix** of A, and the number $\det(M_{rs})$ is the minor of the ***(r,s)th*** entry, a_{rs}. In addition, the numbers

$$A_{ij} = (-1)^{i+j}\det(M_{ij})$$

are called **cofactors** (or **signed minors**).

Example1: Determine the minor matrices M_{11}, M_{23}, and M_{32} for the matrix A given by

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 3 & -3 \\ 4 & 5 & 1 \end{bmatrix}.$$

Definition6.2.2: Let A=(a_{ij}) be an **($n\times n$)** matrix. Then the determinant of A is

$$\det(A) = a_{11}A_{11} + a_{12}A_{12} + \cdots + a_{1n}A_{1n};$$
$$\det(A) = a_{i1}A_{i1} + a_{i2}A_{i2} + \cdots + a_{in}A_{in};$$
$$\det(A) = a_{1j}A_{1j} + a_{2j}A_{2j} + \cdots + a_{nj}A_{nj};$$

where A_{ij} is the cofactor of a_{ij}.

Example2: Compute **det(A)**, where

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 3 & -3 \\ 4 & 5 & 1 \end{bmatrix}.$$

Example3: Compute the determinant of the lower-triangular matrix T, where

$$T = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 2 & 3 & 2 & 0 \\ 1 & 4 & 5 & 1 \end{bmatrix}.$$

Theorem6.2.1: Let $T=(t_{ij})$ be an $(n\times n)$ lower-triangular matrix. Then the determinant of T is

$$\det(T) = t_{11}t_{22}t_{33}\cdots t_{nn}.$$

Obviously $\det(I) = 1.$

6.2 Exercise: P_{454} 21,34

6.3 Elementary operations and determinants

1. Elementary operations

Theorem6.3.1: Let $A=(a_{ij})$ be an $(n\times n)$ matrix, then

$$\det(A^T)=\det(A).$$

$$D=\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = D^T = \begin{vmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22} & \cdots & a_{n2} \\ \cdots & \cdots & \cdots & \cdots \\ a_{1n} & a_{2n} & \cdots & a_{nn} \end{vmatrix}$$

Theorem6.3.2: Let $A=(a_{ij})$ be an $(n\times n)$ matrix. If B is obtained from A by interchanging two columns (or rows) of A, then

$$\det(B) = -\det(A).$$

$$D = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \cdots & \cdots & \cdots & \cdots \\ a_{j1} & a_{j2} & \cdots & a_{jn} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = -D' = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{j1} & a_{j2} & \cdots & a_{jn} \\ \cdots & \cdots & \cdots & \cdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

Theorem6.3.3: Let $A=(a_{ij})$ be an $(n\times n)$ matrix and B is the $(n\times n)$ matrix resulting from multiplying the ith row (or column) of A by a scalar k, then

$$\det(B) = k\det(A).$$

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \cdots & \cdots & \cdots & \cdots \\ ka_{i1} & ka_{i2} & \cdots & ka_{in} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = k \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

Corollary: Let $A=(a_{ij})$ be an $(n\times n)$ matrix and let k be a scalar. Then

$$\det(kA) = k^n\det(A).$$

Theorem6.3.4: Let A,B,C are$(n\times n)$ matrices that are equal except that the ith row (or column) of A is equal to the sum of the ith row (or column) of B and C, then

$$\det(A)=\det(B)+\det(C).$$

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{i1}+b_{i1} & a_{i2}+b_{i2} & \cdots & a_{in}+b_{in} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \cdots & \cdots & \cdots & \cdots \\ b_{i1} & b_{i2} & \cdots & b_{in} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

Theorem6.3.5: Let *A* be an **(*n*×*n*)** matrix, and if a multiple of the *i*th row (or column) is added to the *j*th row (or column), then the determinant is not changed.

$$D = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \cdots & \cdots & \times k & \cdots \\ a_{j1} & a_{j2} & \cdots & a_{jn} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \cdots & \cdots & \cdots & \cdots \\ a_{j1} + ka_{i1} & a_{j2} + ka_{i2} & \cdots & a_{jn} + ka_{in} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

Corollary: Let *A* be an **(*n*×*n*)** matrix, and if the *i*th row (or column) is a multiple of the *j*th row (or column) of *A*, then the determinant is zero.

Theorem6.3.6: ***A*** is an **(*n*×*n*)** singular matrix if and only if the determinant of ***A*** is zero.

Theorem6.3.7: If ***A*** and ***B*** are **(*n*×*n*)** matrices, then

$$\det(AB) = \det(A)\det(B).$$

Theorem6.3.8: If the **(*n*×*n*)** matrix ***A*** is nonsingular, then

$$\det(A) \neq 0, \quad \text{and} \quad \det(A^{-1}) = \frac{1}{\det(A)}.$$

2. Calculate determinants by using properties

(1) Object: Transform matrix to upper(or lower)-triangular matrix by using elementary operation;

(2) Instrument: Creating 1 and 0 by using properties of determinants;

(3) Principle: Elementary operation and properties of determinants.

Example1: Calculate determinant

Solution:

$$D=\begin{vmatrix}4&1&2&5\\1&2&0&2\\10&5&2&0\\1&1&1&7\end{vmatrix}$$

$$D=-\begin{vmatrix}1&1&1&7\\1&2&0&2\\10&5&2&0\\4&1&2&5\end{vmatrix}=-\begin{vmatrix}1&1&1&7\\0&1&-1&-5\\0&-5&-8&-70\\0&-3&-2&-23\end{vmatrix}$$

$\times(-1)$ $\times(-10)$ $\times(-4)$ $\times 5$ $\times 3$

$$= -\begin{vmatrix} 1 & 1 & 1 & 7 \\ 0 & 1 & -1 & -5 \\ 0 & 0 & -13 & -95 \\ 0 & 0 & -5 & -38 \end{vmatrix} \quad (\times(-2))$$

$$= -\begin{vmatrix} 1 & 1 & 1 & 7 \\ 0 & 1 & -1 & -5 \\ 0 & 0 & -3 & -19 \\ 0 & 0 & -5 & -38 \end{vmatrix} \quad (\times(-2))$$

$$= -\begin{vmatrix} 1 & 1 & 1 & 7 \\ 0 & 1 & -1 & -5 \\ 0 & 0 & -3 & -19 \\ 0 & 0 & 1 & 0 \end{vmatrix} \quad (\times 3,\ \times(-1))$$

$$= -\begin{vmatrix} 1 & 1 & 0 & 7 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 0 & -19 \\ 0 & 0 & 1 & 0 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 1 & 7 \\ 0 & 1 & -1 & -5 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -19 \end{vmatrix} = -19$$

Example2:

$$D = \begin{vmatrix} 1991 & 1992 & 1993 \\ 1994 & 1995 & 1996 \\ 1997 & 1998 & 1999 \end{vmatrix}$$

$\times(-1)$ $\times(-1)$

Solution:

$$D = \begin{vmatrix} 1991 & 1 & 1 \\ 1994 & 1 & 1 \\ 1997 & 1 & 1 \end{vmatrix} = 0$$

Example3:

Solution:

$$D = \begin{vmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \\ 4 & 1 & 2 & 3 \end{vmatrix} = \begin{vmatrix} 10 & 10 & 10 & 10 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \\ 4 & 1 & 2 & 3 \end{vmatrix}$$

$$=10\begin{vmatrix}1&1&1&1\\2&3&4&1\\3&4&1&2\\4&1&2&3\end{vmatrix}=10\begin{vmatrix}1&1&1&1\\0&1&2&-1\\0&1&-2&-1\\0&-3&-2&-1\end{vmatrix}$$

$\times(-2)$, $\times(-3)$, $\times(-4)$; $\times(-1)$, $\times 3$

$$=10\begin{vmatrix}1&1&1&1\\0&1&2&-1\\0&0&-4&0\\0&0&4&-4\end{vmatrix}=10\begin{vmatrix}1&1&1&1\\0&1&2&-1\\0&0&-4&0\\0&0&0&-4\end{vmatrix}=160$$