$$\mathcal{L}(1)=\frac{1}{s}$$

$$\mathcal{L}(u(t))=\frac{1}{s}$$

$$\mathcal{L}(\delta(t))=1$$

$$\mathcal{L}(e^{at})=\frac{1}{s-a}$$

$$\mathcal{L}(t^m)=\frac{m!}{s^{m+1}}$$

$$\mathcal{F}(1)=2\pi\delta(w)$$

$$\mathcal{F}(u(t))=\pi\delta(w)+\frac{1}{iw}$$

$$\mathcal{F}(\delta(t))=1$$

$$\mathcal{F}(e^{iat})=2\pi\delta(w-a)$$

$$\mathcal{F}(t^m)=2\pi i^m\delta^{(m)}(w)$$

$$\mathcal{F}(\cos at) = \pi(\delta(w+a) + \delta(w-a))$$

$$\mathcal{F}(\sin at) = i\pi(\delta(w+a) - \delta(w-a))$$

$$\mathcal{L}(\sin at) = \frac{a}{s^2 + a^2}$$

$$\mathcal{L}(\cos at) = \frac{s}{s^2 + a^2}$$

P51 EX5. 求下列函数的付利叶变换:

1) $f(t)=\sin w_0 t \bullet u(t)$

解: $\mathcal{F}\left(\sin w_0 t \bullet u(t)\right)=\mathcal{F}\left(\frac{e^{iw_0 t}-e^{-iw_0 t}}{2i} \bullet u(t)\right)$

$$=\frac{1}{2i}\{\mathcal{F}\left(e^{iw_0 t} \bullet u(t)\right)-\mathcal{F}\left(e^{-iw_0 t} \bullet u(t)\right)\}$$

$$=\frac{1}{2i}\{\frac{1}{iw}+\pi\delta(w)\}\Big|_{w\to w-w_0}-\frac{1}{2i}\{\frac{1}{iw}+\pi\delta(w)\}\Big|_{w\to w+w_0}$$

$$=\frac{w_0}{w_0{}^2-w^2}+\frac{\pi i}{2}\left(\delta(w+w_0)-\delta(w-w_0)\right)$$

$$2)\quad f(t)=e^{-\beta t}\sin w_0 t\bullet u(t)$$

解： $\mathcal{F}\left(e^{-\beta t}\sin w_0 t\bullet u(t)\right)=$

$$=\int_{-\infty}^{+\infty} e^{-\beta t}\sin w_0 t\bullet u(t)e^{-iwt}dt$$

$$=\int_{0}^{+\infty}\sin w_0 t e^{-(\beta+iw)t}dt$$

$$=\mathcal{L}\left(\sin w_0 t\right)\Big|_{s=\beta+iw}$$

$$=\frac{w_0}{s^2+w_0^2}\Big|_{s=\beta+iw}=\frac{w_0}{(\beta+iw)^2+w_0^2}$$

3) $f(t)=e^{-\beta t}\cos w_0 t\bullet u(t)$

解: $\mathcal{F}\left(e^{-\beta t}\cos w_0 t\bullet u(t)\right)=$

$$=\int_{-\infty}^{+\infty} e^{-\beta t}\cos w_0 t\bullet u(t) e^{-iwt}dt$$

$$=\int_0^{+\infty}\cos w_0 t\; e^{-(\beta+iw)t}dt$$

$$=\mathcal{L}(\cos w_0 t)\big|_{s=\beta+iw}$$

$$=\frac{\beta+iw}{s^2+w_0^2}\Big|_{s=\beta+iw}=\frac{\beta+iw}{(\beta+iw)^2+w_0^2}$$

5) $f(t)=e^{iw_0t}u(t-t_0)$

解：$\mathcal{F}\left(e^{iw_0t}u(t-t_0)\right)=\mathcal{F}\left(u(t-t_0)\right)\Big|_{w\to w-w_0}$

$$=e^{-it_0w}\left(\frac{1}{iw}+\pi\delta(w)\right)\Big|_{w\to w-w_0}$$

$$=e^{-it_0(w-w_0)}\left(\frac{1}{i(w-w_0)}+\pi\delta(w-w_0)\right)$$

6) $f(t)=e^{iw_0 t}tu(t)$

解：$F\left(e^{iw_0 t}tu(t)\right)=F\left(tu(t)\right)|_{w\to w-w_0}$

$$=i\left(\frac{1}{iw}+\pi\delta(w)\right)'|_{w\to w-w_0}$$

$$=\left(\frac{-1}{(w-w_0)^2}+\pi i\delta'(w-w_0)\right)$$

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1.(6) $\mathcal{L}[f(t)] = 5\mathcal{L}[\sin 2t] - 3\mathcal{L}[\cos 2t]$

$$= 5\times\frac{2}{s^2+4} - 3\times\frac{s}{s^2+4}$$

1.(8) $\mathcal{L}[f(t)] = \mathcal{L}[e^{-4t}\cos 4t] = \mathcal{L}[\cos 4t]\big|_{s\to s+4}$

$$= \frac{s}{s^2+16}\Big|_{s\to s+4} = \frac{s+4}{(s+4)^2+16}$$

1.(11) $f(t) = u(1-e^{-t})$

解：$u(1-e^{-t}) = u(t)$

$$\mathcal{L}(f(t)) = \mathcal{L}(u(t)) = \frac{1}{s}$$

2. 若 $\mathscr{L}[f(t)]=F(s)$，a 为正实数，证明（相似性质）

$$\mathscr{L}[f(at)]=\frac{1}{a}F\left(\frac{s}{a}\right),$$

(1) 已知 $\mathscr{L}\left[\frac{\sin t}{t}\right]=\operatorname{arctan}\frac{1}{s}$，求 $\mathscr{L}\left[\frac{\sin at}{t}\right]$；

(2) 求 $\mathscr{L}[f(at-b)u(at-b)]$，$b$ 为正实数；

2.(1)因为 $L\left[\frac{\sin t}{t}\right]=\arctan\frac{1}{s}$，所以由相似性质，有

$$L\left[\frac{\sin at}{at}\right]=\frac{1}{a}\arctan\frac{1}{\frac{s}{a}},$$

即 $\frac{1}{a}L\left[\frac{\sin at}{t}\right]=\frac{1}{a}\arctan\frac{a}{s},$

所以 $L\left[\frac{\sin at}{t}\right]=\arctan\frac{a}{s}$

$$L(f(at-b)u(at-b))=$$

解：设$L(f(t))=F(s)$

$$L(f(at-b)u(at-b))$$

$$=\frac{1}{a}L(f(t-b)\,u(t-b))|_{s\to\frac{s}{a}}$$

$$=\frac{1}{a}(e^{-bs}F(s))|_{s\to\frac{s}{a}}=\frac{1}{a}(e^{-\frac{bs}{a}}F(\frac{s}{a}))$$

2.(4)因为 $L[f(t)] = F(s)$, 由相似性质，有

$$L\left[f\left(\frac{t}{a}\right)\right] = aF(as)$$

在利用位移性质，

$$L\left[e^{-at} f\left(\frac{t}{a}\right)\right] = aF(as)\Big|_{s \to s+a}$$

$$= aF(a(s+a))$$

3.

(1) $f(t)=te^{-3t}\sin 2t$,求 $F(s)$;

(2) $f(t)=t\int_0^t e^{-3t}\sin 2t\,dt$,求 $F(s)$;

(3) $F(s)=\ln\dfrac{s+1}{s-1}$,求 $f(t)$;

(1)因为（由位移性质）$L\left[e^{-3t}\sin 2t\right]=\frac{2}{(s+3)^2+4}$

所以利用像函数的微分性质，有

$$L\left[te^{-3t}\sin 2t\right]=-\frac{d}{ds}\left[\frac{2}{(s+3)^2+4}\right]=\frac{4(s+3)}{\left[(s+3)^2+4\right]^2}$$

(2)由积分性质，$L\left[\int_0^t e^{-3t}\sin 2t\,dt\right]$

$$=\frac{1}{s}L\left[e^{-3t}\sin 2t\right]=\frac{1}{s}\frac{2}{(s+3)^2+4}$$

所以 $L\left[t\int_0^t e^{-3t}\sin 2t\,dt\right]=$

$$-\frac{d}{ds}\left[\frac{1}{s}\frac{2}{(s+3)^2+4}\right]=\frac{1}{s^2}\frac{2\left(3s^2+12s+13\right)}{\left[(s+3)^2+4\right]^2}$$

(3) $f(t) = -\frac{1}{t}L^{-1}\left[F'(s)\right]$,

所以 $f(t) = -\frac{1}{t}L^{-1}\left[\frac{d}{ds}\left(\ln\frac{s+1}{s-1}\right)\right]$

$= -\frac{1}{t}L^{-1}\left[\frac{1}{s+1} - \frac{1}{s-1}\right] = -\frac{1}{t}\left(e^{-t} - e^{t}\right)$,

(4)由积分性质，$L\left[\int_0^t te^{-3t}\sin 2t\,dt\right]$

$= \frac{1}{s}L\left[te^{-3t}\sin 2t\right] = \frac{1}{s}\frac{4(s+3)}{\left[(s+3)^2+4\right]^2}$

4. 若 $\mathscr{L}[f(t)]=F(s)$，证明(象函数的积分性质)

$$\mathscr{L}\left[\frac{f(t)}{t}\right]=\int_s^{\infty}F(s)\mathrm{d}s,\text{或 } f(t)=t\mathscr{L}^{-1}\left[\int_s^{\infty}F(s)\mathrm{d}s\right].$$

(1) $f(t)=\dfrac{\sin kt}{t}$，求 $F(s)$；

(2) $f(t)=\dfrac{\mathrm{e}^{-3t}\sin 2t}{t}$，求 $F(s)$；

(3) $F(s)=\dfrac{s}{(s^2-1)^2}$，求 $f(t)$；

(4) $f(t)=\displaystyle\int_0^t\frac{\mathrm{e}^{-3t}\sin 2t}{t}\mathrm{d}t$，求 $F(s)$.