Lecture 8

Stability Criterion

Instructor's Name:
Professor Dr. Hu Guoqing
&
Dr. Jahangir Alam

Outline

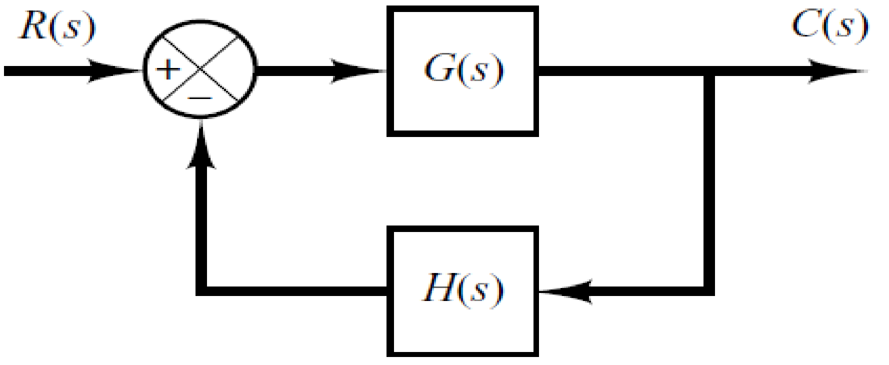
- Transient Response of Higher-Order Systems,
- Real poles and pairs of complex-conjugate poles,
- Dominant Closed-Loop Poles,
- Stability Analysis,
- Stability Analysis in Complex Plane,
- Routh's Stability Criterion
- Relative Stability Analysis
- Application of Routh's Stability Criterion

Transient Response of Higher-Order Systems

Transient Response of Higher-Order Systems

➢The closed-loop transfer function is,

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)}$$



➢Ratios of polynomials in s, where: $G(s) = \frac{p(s)}{q(s)}, \quad H(s) = \frac{n(s)}{d(s)}$

$$\Rightarrow \frac{C(s)}{R(s)} = \frac{\frac{p(s)}{q(s)}}{1+\frac{p(s)}{q(s)} \cdot \frac{n(s)}{d(s)}} = \frac{p(s)d(s)}{q(s)d(s)+p(s)n(s)}$$

$$\Rightarrow \frac{C(s)}{R(s)} = \frac{b_0 s^m + b_1 s^{m-1} + \ldots + b_{m-1} s + b_m}{a_0 s^n + a_1 s^{n-1} + \ldots + a_{n-1} s + a_n}$$

High Order Response behavior of the system to a unit-step input

numerator and the denominator have been factored

$$\frac{C(s)}{R(s)}=\frac{b_0 s^m + b_1 s^{m-1} + \ldots + b_{m-1}s + b_m}{a_0 s^n + a_1 s^{n-1} + \ldots + a_{n-1}s + a_n} \Rightarrow \frac{C(s)}{R(s)}=\frac{K(s+z_1)(s+z_2)\ldots(s+z_m)}{(s+p_1)(s+p_2)\ldots(s+p_n)}$$

$$\Rightarrow C(s)=\frac{a}{s}+\sum_{i=1}^{n}\frac{a_i}{s+p_i}$$

a_i: Residue of the pole at $s=-p_i$

p_i: pole

➢ C(s) Must be expanded, if all closed-loop poles lie in the left-half s plane;

Characteristics of High Order Response

- If there is a closed-loop zero close to a closed-loop pole, then the residue at this pole is small and the coefficient of the transient-response term corresponding to this pole becomes small;

- A pair of closely located poles and zeros will effectively cancel each other;

- If a pole is located very far from the origin, the residue at this pole may be small;

Characteristics of High Order Response

- The transients corresponding to such a remote pole are small and last a short time;

- Terms in the expanded form of C(s) having very small residues contribute little to the transient response, and it may be neglected;

- If this is done, the higher-order system may be approximated by a lower-order one;

Real poles and pairs of complex-conjugate poles

Real poles and pairs of complex-conjugate poles

From, $$C(s)=\frac{a}{s}+\sum_{i=1}^{n}\frac{a_i}{s+p_i}$$

$$C(s)=\frac{a}{s}+\sum_{j=1}^{q}\frac{a_j}{s+p_j}+\sum_{k=1}^{r}\frac{b_k\left(s+\zeta_k\omega_k\right)+c_k\omega_k\sqrt{1-\zeta_k^2}}{s^2+2\zeta_k\omega_k s+\omega_k^2}$$

$$Here,\ q+2r=n$$

- Assumed all closed-loop poles are distinct;
- The response of a higher-order system is composed of first- and second-order systems;

Real poles and pairs of complex-conjugate poles

- The inverse Laplace transform of C(s), The unit-step response c(t) becomes,

$$C(t) = a + \sum_{j=1}^{q} a_j e^{-p_j t} + \sum_{k=1}^{r} b_k e^{-\zeta_k \omega_k t} \cos \omega_k \sqrt{1-\zeta_k^2}\, t$$

$$+ \sum_{k=1}^{r} c_k e^{-\zeta_k \omega_k t} \sin \omega_k \sqrt{1-\zeta_k^2}\, t, \qquad for \;\; t \geq 0$$

- A pair of complex-conjugate poles yields a 2nd-order term in s;
- Involved the responses of first- & 2nd-order systems;

Real poles and pairs of complex-conjugate poles

- Response curve of a stable higher-order system is the sum of a number of exponential curves and damped sinusoidal curves;

$$C(t) = a + \sum_{j=1}^{q} a_j e^{-p_j t} + \sum_{k=1}^{r} b_k e^{-\zeta_k \omega_k t} \cos \omega_k \sqrt{1-\zeta_k^2}\, t$$

$$+ \sum_{k=1}^{r} c_k e^{-\zeta_k \omega_k t} \sin \omega_k \sqrt{1-\zeta_k^2}\, t, \qquad for \;\; t \geq 0$$

- If all closed-loop poles lie in the left-half s plane, then the exponential terms and the damped exponential terms will approach zero as time *t increases, then* steady-state output $C(\infty) = a$;

Real poles and pairs of complex-conjugate poles

- If the system is a stable, then the closed-loop poles are located far from the $j\omega$ axis have large negative real parts.
- The exponential terms correspond to these poles decay very rapidly to zero.
- The horizontal distance from a closed-loop pole to the $j\omega$ axis determines the settling time of transients due to that pole.
- The smaller the distance is, the longer the settling time.

Real poles and pairs of complex-conjugate poles

- The type of transient response is determined by the closed-loop poles;
- The shape of the transient response is primarily determined by the closed-loop zeros;
- The poles of C(s)/R(s) enter into the exponential transient-response terms and/or damped sinusoidal transient-response terms;
- The zeros of C(s)/R(s) do not affect the exponents in the exponential terms, but they do affect the magnitudes and signs of the residues.

Dominant Closed-Loop Poles

- The relative dominance is determined by the ratio of the real parts of the closed-loop poles;
- The relative magnitudes of the residues evaluated at the closed-loop poles;
- The magnitudes of the residues depend on both the closed-loop poles and zeros;
- If the ratios of the real parts of the closed-loop poles exceed 5 and there are no zeros nearby, then the closed-loop poles nearest the $j\omega$ axis will dominate in the transient-response behavior (called *dominant closed-loop poles)* because of decaying slowly.

Dominant Closed-Loop Poles

- The dominant closed-loop poles occur in the form of a complex-conjugate pair;
- The dominant closed-loop poles are most important among all closed-loop poles;
- The gain of a higher-order system is often adjusted a pair of dominant complex-conjugate closed-loop poles.
- The presence of such poles in a stable system reduces the effects of such nonlinearities as dead zone, backlash, and coulomb-friction.

Stability Analysis

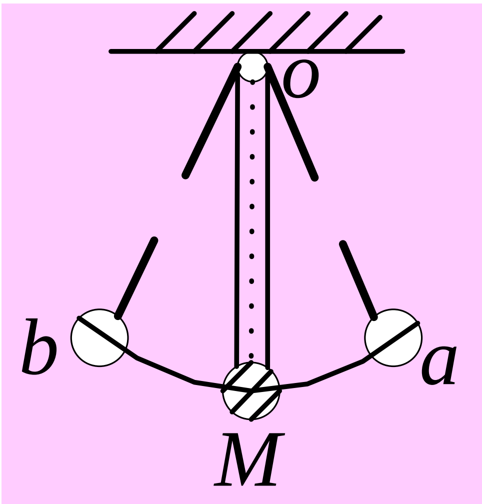
Why investigate stability?

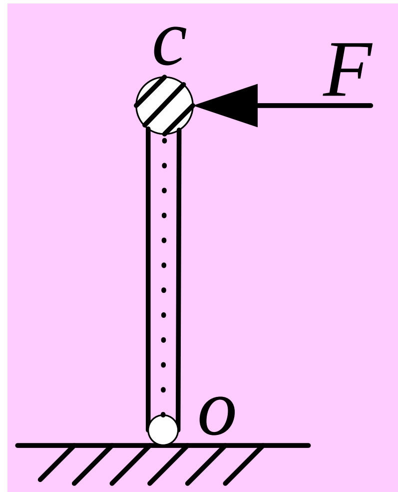
The issue of ensuring the stability of a closed-loop system is the most important to control system design. An unstable feedback system is of no practical value.

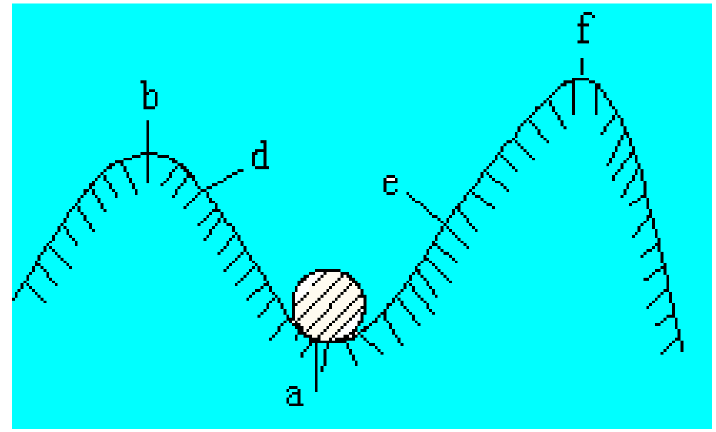
Concept of Stability

- A LTI control system is in equilibrium, and it will departure(偏离) from its equilibrium when it is subjected to transient disturbance;
- A LTI control system is ***stable*** if the output eventually comes back to its equilibrium state when the system is subjected an initial condition;
- A LTI control system is ***critically stable*** if oscillations of the output continue forever;
- It is ***unstable*** if the output diverges without bound from its equilibrium state with initial condition of the system.

Concept of Stability





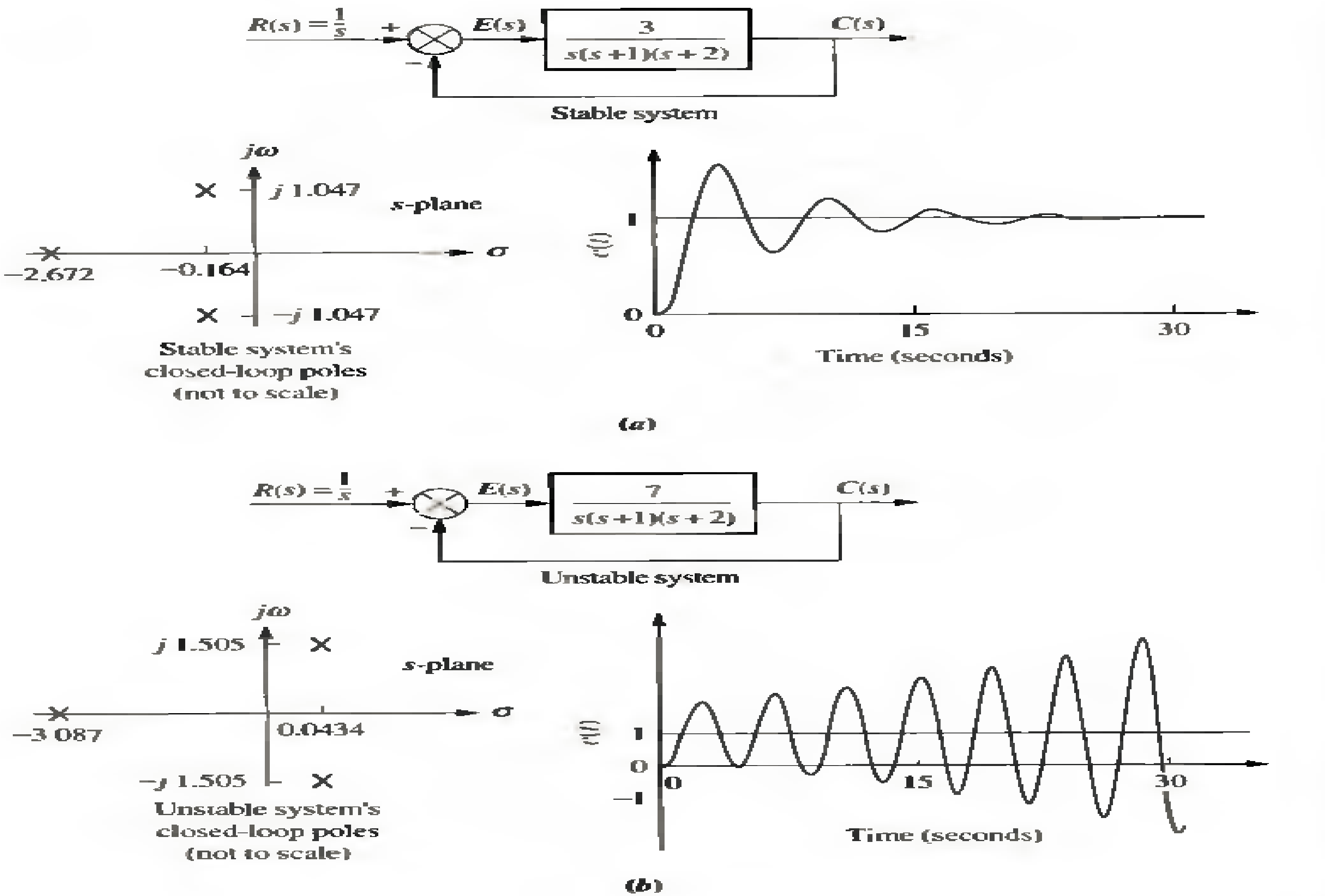


The balance of a pendulum

The balance of a small ball

A **necessary** and **sufficient** condition for a feedback system to be stable is that all the poles of the system transfer function have ***negative real parts***.(闭环特征方程式的根须都位于S的左半平面)

Example: Stable & Unstable System



Stability Analysis in Complex Plane

- Stability is the most important problem in linear control systems;
- what conditions will a system become unstable?
- If it is unstable, how stabilize the system?
- Stable iff all closed-loop poles lie in the left-half s plane;

Stability Analysis in Complex Plane

- The stability of a linear closed-loop system can be determined from the location of the closed-loop poles in the s plane;
- If any of these poles lie in the right-half s plane, then with increasing time they give rise to the dominant mode;
- The transient response increases monotonically or oscillates with increasing amplitude. This represents an unstable system;

Stability Analysis in Complex Plane

- The power is turned on, the output may increase with time;
- If no saturation takes place in the system and no mechanical stop is provided, then the system may eventually be subjected to damage and fail, since the response of a real physical system cannot increase indefinitely.
- The closed-loop poles in the right-half s plane are not permissible in the usual linear control system.
- If all closed-loop poles lie to the left of the $j\omega$ axis, any transient response eventually reaches equilibrium. This represents a stable system.

Stability Analysis in Complex Plane

- A linear system is stable or unstable is a property of the system itself and does not depend on the input or driving function of the system.
- The poles of the input, or driving function, do not affect the property of stability of the system, but they contribute only to steady-state response terms in the solution.
- The problem of absolute stability can be solved readily by choosing no closed-loop poles in the right-half s plane, including the jω axis.

Stability Analysis in Complex Plane

- Mathematically, closed-loop poles on the $j\omega$ axis will yield oscillations, the amplitude of which is neither decaying nor growing with time;
- If noise is present, the amplitude of oscillations may increase at a rate determined by the noise power level;
- Therefore, a control system should not have closed-loop poles on the $j\omega$ axis.