Partial Differential Equations and Waves

Lecture 7: 1D Wave Equation (Part 3) and Water Waves

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Before we plunge into the water… …let's have a second look at waves

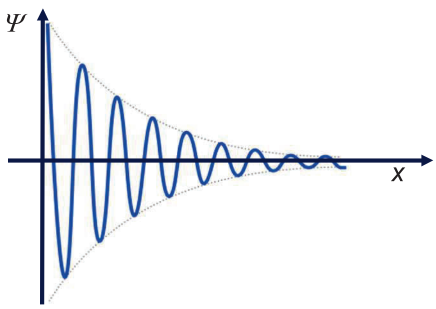
- We continue our previous lecture on waves by considering:
  - *Lossy (dispersive) waves*
  - *Evanescent waves*
  - *Amplitude modulation*

Lossy (damped) wave

- Let us consider a damped (or lossy) wave, whose amplitude decays as the wave propagates:

$$A(x) = Ae^{-lx}$$

  - where *l* is the “loss factor”



Complex wave number

- This leads to a decaying waveform:

$$\psi(x,t) = A(x)\,e^{j(\omega t - kx)} = Ae^{-lx}\,e^{j(\omega t - kx)}$$
$$= Ae^{j(\omega t - (k - jl)x)} = Ae^{j(\omega t - \kappa x)}$$

- This means that, for a lossy wave, the wave number becomes complex:

$$\kappa = (k - jl)$$

Example: 1D string in a viscous fluid

- We introduced the wave equation by studying the 1D string.
- We now re-visit the string, but this time place it in a viscous medium so that it loses energy to the medium.
- This causes damped vibrations, familiar to you from the first year.

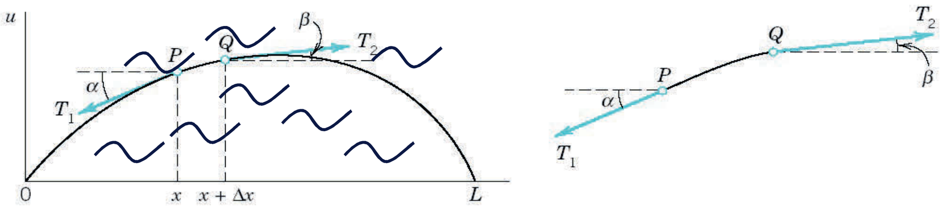


Fig. 283. Deflected string at fixed time *t*. Explanation on p. 539 *Kreyszig*

Damped wave equation

- Re-doing the analysis that led to the wave equation, the equilibrium equation becomes:

$$\mu \Delta x \frac{\partial^2 \psi}{\partial t^2} = T \Delta x \frac{\partial^2 \psi}{\partial x^2} \boxed{- \beta \frac{\partial \psi}{\partial t} \Delta x,}$$

Damping term

where the term $\beta \frac{\partial \psi}{\partial t}$ corresponds to viscous damping and β denotes the resistance per unit length.

- This we can rearrange to find:

$$\frac{\partial^2 \psi}{\partial t^2} + \frac{\beta}{\mu}\frac{\partial \psi}{\partial t} = \frac{T}{\mu}\frac{\partial^2 \psi}{\partial x^2} \quad \text{or} \quad \frac{\partial^2 \psi}{\partial t^2} + \frac{\beta}{\mu}\frac{\partial \psi}{\partial t} = c^2 \frac{\partial^2 \psi}{\partial x^2}$$

- Or more generally: $\boxed{\frac{\partial^2 \psi}{\partial t^2} + \Gamma \frac{\partial \psi}{\partial t} = c^2 \frac{\partial^2 \psi}{\partial x^2}}$

Dispersion relation for a damped wave

- Substituting into this equation the general solution for a forward travelling wave,

$$\psi(x,t) = A\,e^{j(\omega t - \kappa x)}$$

  - (recalling that we now have a complex wave number κ)
- We obtain:

$$\frac{\partial^2 \psi}{\partial t^2} + \Gamma \frac{\partial \psi}{\partial t} = c^2 \frac{\partial^2 \psi}{\partial x^2}$$

$$\psi_t(x,t) = j\omega\psi(x,t)$$

$$\psi_{tt}(x,t) = -\omega^2\psi(x,t)$$

$$\psi_{xx}(x,t) = -\kappa^2\psi(x,t)$$

$$\Rightarrow -\omega^2\psi + \Gamma\, j\omega\,\psi = -c^2\kappa^2\psi$$

- From this we find a complex dispersion relation:

$$\boxed{\omega^2 - \Gamma\, j\omega = c^2\kappa^2}$$

Analysing the dispersion relation

- We substitute the complex wave number $\kappa = (k - jl)$ into the dispersion relation:

$$\omega^2 - \Gamma\, j\omega = c^2\kappa^2 \quad \Rightarrow \quad \omega^2 - j\,\Gamma\,\omega = c^2\left(k^2 - l^2 - 2\,j\,k\,l\right)$$

  - Equating real and imaginary parts:

$$\omega^2 = c^2\left(k^2 - l^2\right) \qquad \Gamma\omega = 2c^2\,kl$$

  - This allowsus to find the loss factor: $\boxed{l = \frac{\Gamma\omega}{2c^2 k} = \frac{1}{2k}\frac{\Gamma}{\omega}\left(\frac{\omega}{c}\right)^2}$
  - Substituting l into the real part:

$$\omega^2 = c^2\left(k^2 - \left(\frac{1}{2k}\frac{\Gamma}{\omega}\left(\frac{\omega}{c}\right)^2\right)^2\right) \quad \Rightarrow \quad 0 = k^4 - \left(\frac{\omega}{c}\right)^2 k^2 - \frac{1}{4}\left(\frac{\Gamma}{\omega}\right)^2\left(\frac{\omega}{c}\right)^4$$

  - Finally yields: $\boxed{k^2 = \frac{1}{2}\left(\frac{\omega}{c}\right)^2\left[1 \pm \sqrt{1 + (\Gamma/\omega)^2}\right]}$

Extreme case: light damping

- For light damping, $\frac{\Gamma}{\omega} = \frac{\beta}{\mu}\frac{1}{\omega} << 1$, leading to:

$$k^2 = \frac{1}{2}\left(\frac{\omega}{c}\right)^2\left[1 \pm \sqrt{1+(\Gamma/\omega)^2}\right] \approx \left(\frac{\omega}{c}\right)^2 \Rightarrow k \approx \pm\frac{\omega}{c}$$

- From which we recover the (now very small) loss factor:

$$l = \frac{\Gamma\omega}{2c^2 k} \approx \pm\frac{1}{2\,\omega/c}\frac{\Gamma}{\omega}\left(\frac{\omega}{c}\right)^2 = \pm\frac{\Gamma}{2c}$$

- Not surprisingly, the situation is almost that of an undamped wave:

$$\psi(x,t) = A\,e^{j(\omega t - \kappa x)} \approx A\,e^{j(\omega t - kx)}$$

  - The only difference is that mild attenuation (decrease of amplitude with distance) occurs with an almost constant attenuation length of $1/|l|$:

$$A(x) = Ae^{-lx} = Ae^{-\frac{\Gamma}{2c}x}$$

Extreme case: heavy damping

- Working from our derived dispersion relationships:

$$l = \frac{\Gamma\omega}{2c^2 k} = \boxed{\frac{1}{2k}\left(\frac{\omega}{c}\right)^2\frac{\Gamma}{\omega}} \qquad k^2 = \frac{1}{2}\left(\frac{\omega}{c}\right)^2\left[1 \pm \sqrt{1+\left(\frac{\Gamma}{\omega}\right)^2}\right]$$

- Heavy damping implies that $\frac{\Gamma}{\omega} >> 1$, so that:

$$\boxed{k^2 \approx \pm\frac{1}{2}\left(\frac{\omega}{c}\right)^2\frac{\Gamma}{\omega}} \Rightarrow \quad k^2 \approx k\boxed{\frac{1}{2k}\left(\frac{\omega}{c}\right)^2\frac{\Gamma}{\omega}} = kl$$

(we can drop the sign as $k^2 > 0$ and k and l always have the same sign)

Clearly, $k^2 \approx kl$ implies that $k \approx l$ from which follows that $\kappa \approx k - jk$

- The now heavily damped wave becomes:

$$\psi(x,t) = A\,e^{j(\omega t - \kappa x)} \approx A\,e^{j(\omega t - (k - jk)x)} = A\,e^{-kx}e^{j(\omega t - kx)}$$

  - Now the attenuation becomes very large, with an attenuation length $\boxed{1/|l| \propto 1/\sqrt{\omega}}$

Group velocity: velocity of a wave group

- We know what the phase velocity is: $v_\phi = \frac{\omega}{k}$
  - i.e. the rate at which the phase of the wave travels in space.
- We haven't yet introduced group velocity:
  - The group velocity of a wave is the velocity with which the overall shape of the wave's amplitudes (its envelope) travels.
  - It is defined as: $v_g = \frac{d\omega}{dk}$
  - The function $\omega(k)$, which gives ω as a function of k, is known as the dispersion relation.
  - If ω is directly proportional to k, e.g. $\omega = \pm ck \Rightarrow d\omega/dk = \pm c$ then the group velocity is exactly equal to the phase velocity, and the wave is non-disper .
  - Otherwise, the individual waves will travel at different speeds, and dispersion (smearing out) will take place.

Phase and group velocity for a vibrating string in a viscous fluid

- Recall the damped wave equation: $\frac{\partial^2\psi}{\partial t^2} + \Gamma\frac{\partial\psi}{\partial t} = c^2\frac{\partial^2\psi}{\partial x^2}$
  - We use a complex wave number: $\kappa = k - jl$
  - And the general dispersion relation: $\omega^2 - j\Gamma\omega = c^2\kappa^2$
  - Equating real parts of the dispersion relation: $\omega^2 = c^2\left(k^2 - l^2\right)$
- By definition, the phase velocity is given by:

$$v_\phi = \frac{\omega}{k} = \pm\frac{\sqrt{c^2(k^2 - l^2)}}{k} = \pm c\sqrt{1 - \left(\frac{l}{k}\right)^2} \quad \text{(where } k \text{ determines the sign and hence the direction)}$$

- We can find the group velocity by:

$$v_g = \frac{d\omega}{dk} = \frac{d}{dk}\left(\pm\sqrt{c^2(k^2 - l^2)}\right) = \pm\frac{ck}{\sqrt{k^2 - l^2}} = \pm\frac{c^2 k}{\omega} = \pm\frac{c}{\sqrt{1 - (l/k)^2}}$$

- From this we find the relation: $v_\phi v_g = c^2$