Modeling & Optimization of Complex Networked Systems: Applications to Operations Scheduling & Supply Network Management

- Reading Assignment: Bertsekas Sections 3.1 and 3.2
- Last Time:
    - Conjugate Direction Methods

  2. OPTIMIZATION OVER A CONVEX SET
    - Necessary and Sufficient Conditions for Optimality
    - Feasible Directions and the Gradient Projection Methods
- Today:
    - Lagrange Multiplier Theory: Necessary Conditions for Equality Constraints
    - The Lagrangian Relaxation Approach
    - Sufficient Conditions and Sensitivity Analysis
- Next Time: Bertsekas Sections 3.3, 3.4, 4.3(?), and 5.1

- Conjugate Direction Methods
  - Making the best use of quadratic properties

  $x^{k+1} = x^k + \alpha^k d^k$, α^k obtained by line minimization

  $d^k = -\nabla f(x^k) + \beta^k d^{k-1}$, with

$$\beta^k = \frac{\nabla f(x^k)'\nabla f(x^k)}{\nabla f(x^{k-1})'\nabla f(x^{k-1})} = \frac{\nabla f(x^k)'\left(\nabla f(x^k) - \nabla f(x^{k-1})\right)}{\nabla f(x^{k-1})'\nabla f(x^{k-1})}$$

- Conditions for Constrained Optimality
  - $\nabla f(x^*)'(x - x^*) \geq 0 \ \forall \ x \in X$

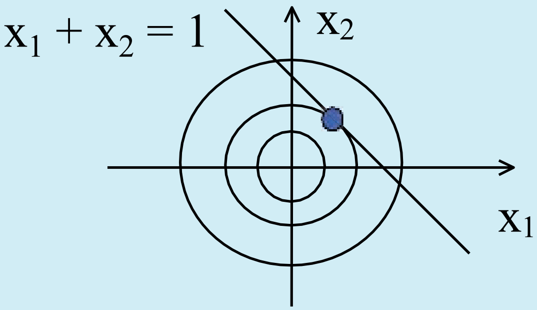
  $\Rightarrow$ Any feasible direction leads to increase in f(x) at least locally
- Feasible Directions and the Gradient Projection Methods
  - Constraints are **always satisfied** ~ the **Primal Approach**
  - If we are on the boundary, **project** negative gradient back onto the set of active constraints
  - With α^k chosen by the Armijo rule or the limited minimization rule, every limit point of $\{x^k\}$ is stationary
  - Convergence similar to the gradient method

Lagrange Multiplier Theory: Necessary Conditions for Equality Constraints

- A "**dual**" approach without requiring the constraints to be satisfied across the iterations
- We shall first present intuitive ideas about necessary conditions, and follow up by more rigorous derivations
- A motivating example

  $\min_x f(x)$, with $f(x) \equiv x_1^2 + x_2^2$, subject to $x_1 + x_2 = 1$

  – What is the problem?
  – What is the solution?
  – How many methods are there to solve the problem?
  – How do they work? Pros and cons?



- **Method 1:** Graphical inspection
- **Method 2:** Direct substitution

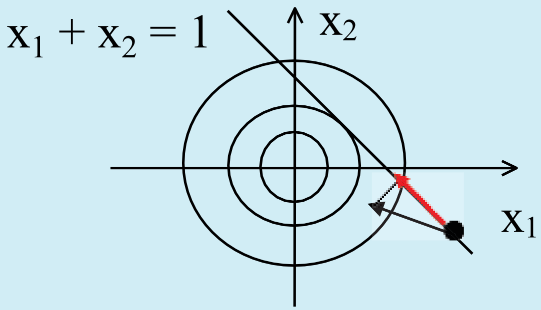
$x_2 = 1 - x_1$

$f(x_1,(1-x_1)) = x_1^2 + (1-x_1)^2 = 2x_1^2 - 2x_1 + 1 \equiv F(x_1)$

$\Rightarrow$ An **unconstrained** optimization

$dF(x_1)/dx_1 = 0 = 4x_1 - 2 \Rightarrow x_1^* = 0.5,\ x_2^* = 1 - 0.5 = 0.5$

- **Method 3:** Gradient projection method



- **Method 4:** Lagrangian relaxation method
- **Method 5:** A two-level iterative LR approach
- These last two methods will be discussed later

- The general problem formulation:

  Minimize f(x), subject to

  $h_i(x) = 0, i = 1, .., m,$

  $g_j(x) \le 0, j = 1, .., r.$

  Or, h(x) = 0 and g(x) ≤ 0

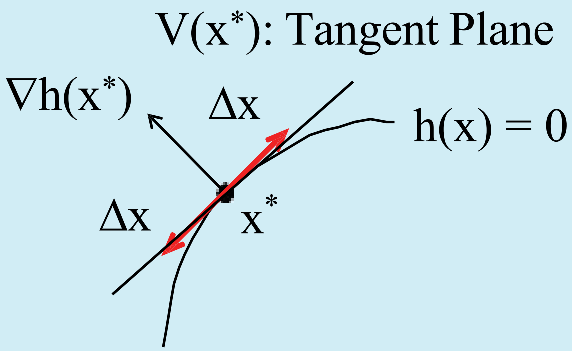
  – Basic assumptions: f(x), $h_i(x)$ and $g_j(x) \in C^1$

  – Most times we shall start with h(x) = 0, then extend the results to include g(x) ≤ 0
- Previously we studied necessary and sufficient conditions, and developed numerical methods to solve the problem
- We will derive a different set of necessary and sufficient conditions based on **Lagrangian relaxation**, and present a series of methods to solve the problem
- We will start with the concept of **tangent plane**, and examine what conditions that x^* has to satisfy

Tangent Planes

- The problem:

  Minimize f(x), subject to $h_i(x) = 0$, i = 1, .., m
- What is a tangent plan at x^*? How to characterize it?



- **Tangent plane** of **h(x) = 0** at x^*:

  $$V(x^*) \equiv \{x \mid \nabla h_i(x^*)' (x - x^*) = 0, i = 1, .., m\}$$

  – It is **characterized by** and **orthogonal to** $\nabla h(x^*)$

  – It indicates possible directions of **infinitesimal move** along the active constraint(s), or the **first order feasible variations**