Model and definitions

commodities (goods) $l = 1, ..., L$
consumers $i = 1, ..., I$

- consumption set $X_i \subset IR^L_+$
- preferences $\;_i$ complete and transitive, defined on X_i

firms $j = 1, ..., J$

- production set $Y_j \subset IR^L$ closed and non-empty
- sign convention: $y_{jl} < 0$ input, $y_{jl} > 0$ output
- profit $p.y_j$

global endowment $\$ \in IR^L_{++}$

notation: $x_i \in IR^L_+$, $x = (x_1, ..., x_I) \in IR^{LI}_+$, $y_j \in IR^L$,
$y = (y_1, ..., y_J) \in IR^{LJ}$, $p = (p_1, ..., p_L)$

Definitions:

feasible allocation: an allocation $(x, y) \subset R^{(I+J)L}$ such that $\sum_i x_i = \$ + \sum_j y_j$

PO allocation: feasible allocation such that there is no other feasible allocation that Pareto-dominates it ($\forall i, x_i' \ {}_i \ x_i$ and $\exists i_0 / x_{i_0}' \ {}_{i_0} \ x_{i_0}$).

Private property economy:

- individual endowment $\omega_i \in I\!R^L_+$ ($\sum_i \omega_i = \$$)
- right to a share of firms' profit $\theta_{ij} \in [0, 1]$ ($\forall j, \sum_i \theta_{ij} = 1$)

+ free trade on markets, optimizing and price-taking behaviour,...

wealth of i: $w_i = p.\omega_i + \sum_j \theta_{ij} p.y_j$

Definition: A Walrasian equilibrium is (x^*, y^*, p) such that

- $\forall j, y_j^*$ maximizes $p.y_j$
- $\forall i, x_i^*$ maximal element of ${}_i$ in the budget set

$$x_i \in X_i / p.x_i \leq p.\omega_i + \sum_j \theta_{ij} p.y_j^*$$

- market clearing: $\sum_i x_i^* = \omega + \sum_j y_j^*$

Definition: A price equilibrium with transfers is (no repartition of the profits and endowment is required) (x^*, y^*, p) such that

- there is $(w_1, \ldots, w_I) \in I\!R^I_+$ such that
$$\sum_i w_i = p.\$ + \sum_j p.y_j^*$$
- $\forall j, y_j^*$ maximizes $p.y_j$
- $\forall i, x_i^*$ maximal element of $_i$ in the budget set
$$\{x_i \in X_i / p.x_i \leq w_i\}$$
- market clearing: $\sum_i x_i^* = \$ + \sum_j y_j^*$

A Walrasian equilibrium is a price equilibrium with transfers (for $w_i = p.\omega_i + \sum_j \theta_{ij} p.y_j^*$)

First Theorem of Welfare Economics

One weak assumption: locally non satiated preferences
$\forall x_i \in X_i, \forall \varepsilon > 0, \exists x \in X_i / \|x_i - x\| < \varepsilon$ and $x \quad x_i$

Two implicit strong assumptions:

- agents are price takers
- complete markets ("any good can be traded against any other good" - a formal definition is given in a next chapter)

Theorem: If consumers' preferences are locally non satiated, then the allocation (x, y) of a price equilibrium with transfers (or a Walrasian equilibrium) is PO.

Proof:

Let (x^*, y^*, p) an equilibrium (associated with a repartition $w_1, \ldots, w_I$ of wealth among consumers)

$\forall i, x_i \ {}_i\, x_i^* \Longrightarrow p.x_i > w_i$
because of preferences maximization
(to increase one consumer's utility requires to give him more wealth)

$\forall i, x_i \ {}_i\, x_i^* \Longrightarrow p.x_i \geq w_i$
because of preferences maximization and local non satiation

One shows the implication "non $B \Rightarrow$ non A" (instead of "$A \Rightarrow B$": if $p.x_i < w_i$, then one has x_i' such that $p.x_i' < w_i$ and $x_i' \ {}_i\, x_i$, hence $x_i^* \ {}_i\, x_i' \ {}_i\, x_i$

(to maintain others' utility requires not to decrease their wealth)

Hence, (this is a proof by contradiction) if there is an allocation (x, y) that Pareto dominates (x^*, y^*), then (x, y) satisfies

$$\sum_i p.x_i > \sum_i w_i = p.\$ + \sum_j p.y_j^*.$$

(the value of $x >$ the value of x^* because of local non satiation)
But $\forall j, p.y_j^* \geq p.y_j$
(it is impossible to increase the value of the available wealth by adjusting production)
Hence

$$\sum_i p.x_i > p.\$ + \sum_j p.y_j,$$

The allocation (x, y) is not feasible: a contradiction.
QED (Quod Erat Demonstrandum).

In an Edgeworth box:

- Example with non convex preferences
- Counter-example with locally satiated preferences (satiation for one agent is enough): one can reduce one's wealth without decreasing his utility (in order to give this wealth to others and increase their utility)

Second Theorem of Welfare Economics

- a bibliographical remark: if you have never heard of the second theorem or if you don't like statements, then don't begin your readings with Mas-Colell et alii!
- Mas-Colell states a version of the second theorem with a concept of quasi-equilibrium (weaker than the equilibrium) to distinguish the role played by the convexity assumption from the role played by other assumptions (these are mainly technical and have barely no economic content - the second theorem stated with the quasi-equilibrium requires fewer technical assumptions)... this makes section 16D hard to read
- the theorem in these slides combines prop 16.1 and 16.3 in Mas-Colell et alii

Theorem:

Assumptions:

- $\forall j$, Y_j convex + technical assumption (free disposal: $Y_j - IR^L_+ \subset Y_j$)
- $\forall i$, $_i$ convex (u_i quasi-concave) and locally non satiated + technical assumptions ($_i$ continuous, $X_i = IR^L_+$)

Let a Pareto optimal allocation (x^*, y^*) s.t. $x^* \quad 0$

There is $p \neq 0$ s.t. (x^*, y^*, p) is a price equilibrium with transfers (associated with $w_i = p.x_i^*$).

Remark about the convexity of a production set Y:

In the case of a single output (where Y is represented by

$y_L \leq f(-y_1, \ldots, -y_{L-1})$ if the output is good L),

$$Y \text{ convex} \Leftrightarrow f \text{ concave}$$

In general:

- If $0 \in Y$ and Y convex, then the returns to scale are decreasing (and can be constant).
- mixing inputs combinations is at least as good as extreme inputs combinations: if y and y' have the same output (the same positive components), then $ay + (1-a)y'$ has the same output as well and belongs to Y... But it is possible that Y contains an element with the same inputs and larger outputs.

Remark: Returns to Scale

- decreasing (DRTS) : $\forall y \in Y, \forall a \in [0, 1], ay \in Y$
- increasing (IRTS) : $\forall y \in Y, \forall a \geq 1, ay \in Y$
- constant (CRTS) : $\forall y \in Y, \forall a \geq 0, ay \in Y$

In the case with a single output, CRTS $\Leftrightarrow$ the production function is homogenous of degree 1

Proof:

scheme of the proof:

1. find p with a ”separating hyperplane” theorem
2. check that (x^*, y^*, p) satisfies the definition of equilibrium

One separates the 2 following sets:

- $Y + \{\$\} = \sum_j Y_j + \{\$\}$: quantities of available goods
- $V_i = \{x_i \in X_i / x_i \succ x_i^*\}$, $V = \sum_i V_i$: quantities of goods allowing for an allocation that Pareto-dominates x^*.

$Y + \{\$\}$, V_i, V convex sets (a sum of convex sets is convex)

(x^*, y^*) PO $\Rightarrow V \cap (Y + \{\$\}) = \emptyset$

By a separation theorem (annexe M.G.), there are $p \in IR^L$ ($p \neq 0$) and $r \in IR$ s.t.

- $\forall z \in V, p.z \geq r$
- $\forall z \in Y + \{\$\}, p.z \leq r$

One checks the definition of equilibrium:

One checks first that r = value of x^*:

- $\sum_i x_i^* = \sum_j y_j^* + \$ \in Y + \{\$\}$ hence $p.\sum_i x_i^* \leq r$
- because of local non satiation, $\forall \varepsilon > 0, \exists x / \|x - x^*\| \leq \varepsilon, \sum_i x_i \in V$ and then $p.\sum_i x_i \geq r$. Continuity implies: $p.\sum_i x_i^* \geq r$.
- hence $r = p.\sum_i x_i^* = p.\sum_j y_j^* + p.\$$.

One checks that y_j^* max $p.y_j$:

$$
\begin{aligned}
y_j + \sum_{k \neq j} y_k^* + \$ &\in Y + \{\$\} \\
\text{hence } p.y_j + p.\sum_{k \neq j} y_k^* + p.\$ &\leq r = p.\sum_{k} y_k^* + p.\$ \\
\text{hence } p.y_j &\leq p.y_j^*
\end{aligned}
$$

One checks that x_i^* is maximal in the budget set, i.e.

$$x_i \succ_i x_i^* \Rightarrow p.x_i > p.x_i^*$$

same idea as for y_j^* but more intricate to write.

- One wants: $p.x_i + p.\sum_{k \neq i} x_k^* \geq r$
- Local non satiation implies: $\forall \varepsilon > 0, \forall k \neq i, \exists x_k / \|x_k - x_k^*\| < \varepsilon, x_i + \sum_{k \neq i} x_k \in V$ and then $p.(x_i + \sum_{k \neq i} x_k) \geq r$. By continuity w.r.t. x_k, $p.x_i + p.\sum_{k \neq i} x_k^* \geq r$
- Conclusion: $p.x_i + p.\sum_{k \neq i} x_k^* \geq p.\sum_k x_k^*$ hence $p.x_i \geq p.x_i^*$.
- By contradiction, one checks that $p.x_i > p.x_i^*$. If $p.x_i = p.x_i^*$, then $\forall a \in]0, 1[$, $p.(ax_i) < p.x_i^*$, which implies $x_i^* \succ_i (ax_i)$. As $x_i \succ_i x_i^*$ and, by continuity of $\succsim_i$, one has $ax_i \succ_i x_i^*$ for a around 1, which is a contradiction.

QED

Remark: the prices exhibited in the theorem are non negative

- If $p_l < 0$, then y_j^* cannot maximize the profit of good l as decreasing y_l (which is possible because of the free disposal assumption) increases the profit.
- this implies that the equilibrium wealth of every agent is positive ($w_i = p.x_i^* > 0$)
- a technical detail: in the proof of Th 2, $w_i > 0$ allows us to write: "$\forall a \in]0, 1[, p.(ax_i) < p.x_i^*$" (see the previous slide where this detail was ignored)

In an Edgeworth box, counter-example with non convex preferences.

Beyond convexity, strong assumptions are required to interpret Th 2 as the decentralization of an optimum by a government:

- agents are price-takers (for example: because gvt able to choose the price by buying/selling any quantity supplied/demanded at the chosen price)
- gvt needs a lot of information about preferences to compute the right prices
- and about individual endowment to compute the transfers

In the real life, taxes are based on variables that can be observed (incomes,...). Furthermore, gvt needs to give to agents appropriate incentives in order to observe these variables. Hence, taxes create distortion in the allocation: one must choose between PO and equity of the allocation.

First Order Conditions for PO

One considers a differentiable problem:

- $\forall i, u_i \; C^2$ on IR^L_+, $u_i(0) = 0$, $\nabla u_i \quad 0$ (i.e. $\forall l, \frac{\partial u_i}{\partial x_l} > 0$), u_i quasi-concave
- $\forall j, Y_j = \; y \in IR^L_+ / F_j(y) \leq 0 \;$, $F_j(0) \leq 0$, $F_j \; C^2$, $\nabla F_j \quad 0$, F_j concave (F_j transformation function, $F_j(y_j) = 0$: transformation frontier)

- one writes the FOC characterizing a PO allocation and the FOC characterizing an equilibrium
- one sees that they have the same solutions
- hence, we have a proof of the 2 Welfare Theorems