

专题 16 数列解答题分类练

一、方程思想求数列通项

1. (2024 届山东省齐鲁名校高三上学期联合检测) 记等比数列 $\{a_n\}$ 的前 n 项和为 S_n , 已知 $a_3, \frac{5}{2}, \frac{7}{2}, S_3$ 成等差数列.

(1) 求 $\{a_n\}$ 的通项公式;

(2) 设 $b_n = (2n-1) \cdot a_n$, 求数列 $\{b_n\}$ 的前 n 项和 T_n .

【解析】(1) 设 $\{a_n\}$ 的公比为 q , 由 $a_3, \frac{5}{2}, \frac{7}{2}, S_3$ 成等差数列, 得 $a_3 = \frac{3}{2}, S_3 = \frac{9}{2}$.

当 $q=1$ 时, $S_3 = 3a_3 = 3 \times \frac{3}{2} = \frac{9}{2}$, 符合题意, 所以 $a_n = \frac{3}{2}$;

当 $q \neq 1$ 时, $\begin{cases} \frac{a_1(1-q^3)}{1-q} = \frac{9}{2}, \\ a_1q^2 = \frac{3}{2}, \end{cases}$ 所以 $a_1 = 6, q = -\frac{1}{2}$, 则 $a_n = 6 \times \left(-\frac{1}{2}\right)^{n-1}$.

综上, $a_n = \frac{3}{2}$ 或 $a_n = 6 \times \left(-\frac{1}{2}\right)^{n-1}$.

(2) 当 $a_n = \frac{3}{2}$ 时, $b_n = (2n-1) \cdot \frac{3}{2}$,

所以 $T_n = \frac{3}{2} \times [1+3+5+\dots+(2n-1)] = \frac{3}{2} \times \frac{n(2n-1+1)}{2} = \frac{3n^2}{2}$;

当 $a_n = 6 \times \left(-\frac{1}{2}\right)^{n-1}$ 时, $b_n = 6(2n-1) \cdot \left(-\frac{1}{2}\right)^{n-1}$,

所以 $T_n = 6 \times \left[1 \times \left(-\frac{1}{2}\right)^0 + 3 \times \left(-\frac{1}{2}\right)^1 + \dots + (2n-1) \times \left(-\frac{1}{2}\right)^{n-1} \right]$,

则 $-\frac{1}{2}T_n = 6 \times \left[1 \times \left(-\frac{1}{2}\right)^1 + 3 \times \left(-\frac{1}{2}\right)^2 + \dots + (2n-1) \times \left(-\frac{1}{2}\right)^n \right]$,

所以

$$\frac{1}{4}T_n = 1 + 2 \times \left[\left(-\frac{1}{2}\right)^1 + \left(-\frac{1}{2}\right)^2 + \dots + \left(-\frac{1}{2}\right)^{n-1} \right] - (2n-1) \times \left(-\frac{1}{2}\right)^n = 1 + 2 \times \frac{-\frac{1}{2} - \left(-\frac{1}{2}\right)^n}{1 - \left(-\frac{1}{2}\right)} - (2n-1) \times \left(-\frac{1}{2}\right)^n$$

$$= \frac{1}{3} - \left(2n + \frac{1}{3}\right) \cdot \left(-\frac{1}{2}\right)^n,$$

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所以 $T_n = \frac{3}{2} \times [1+3+5+\dots+(2n-1)] = \frac{3}{2} \times \frac{n(2n-1+1)}{2} = \frac{3n^2}{2}$;

当 $a_n = 6 \times \left(-\frac{1}{2}\right)^{n-1}$ 时, $b_n = 6(2n-1) \cdot \left(-\frac{1}{2}\right)^{n-1}$,

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则 $-\frac{1}{2}T_n = 6 \times \left[1 \times \left(-\frac{1}{2}\right)^1 + 3 \times \left(-\frac{1}{2}\right)^2 + \dots + (2n-1) \times \left(-\frac{1}{2}\right)^n \right]$,

所以

$$\frac{1}{4}T_n = 1 + 2 \times \left[\left(-\frac{1}{2}\right)^1 + \left(-\frac{1}{2}\right)^2 + \dots + \left(-\frac{1}{2}\right)^{n-1} \right] - (2n-1) \times \left(-\frac{1}{2}\right)^n = 1 + 2 \times \frac{-\frac{1}{2} - \left(-\frac{1}{2}\right)^n}{1 - \left(-\frac{1}{2}\right)} - (2n-1) \times \left(-\frac{1}{2}\right)^n$$

$$= \frac{1}{3} - \left(2n + \frac{1}{3}\right) \cdot \left(-\frac{1}{2}\right)^n,$$

$$\text{所以 } T_n = \frac{4}{3} - \left(2n + \frac{1}{3}\right) \cdot \left(-\frac{1}{2}\right)^{n-2}.$$

$$\text{综上, } T_n = \frac{3n^2}{2} \text{ 或 } T_n = \frac{4}{3} - \left(2n + \frac{1}{3}\right) \cdot \left(-\frac{1}{2}\right)^{n-2}$$

2. (2023 届天津市宁河区芦台第一中学高三上学期期末) 已知数列 $\{a_n\}$ 是公差为 1 的等差数列, 且 $a_1 + a_2 = a_3$,

数列 $\{b_n\}$ 是等比数列, 且 $b_1 \cdot b_2 = b_3, a_4 = 4b_1 - b_2$.

(1) 求 $\{a_n\}$ 和 $\{b_n\}$ 的通项公式;

(2) 令 $d_n = \frac{b_{n+1}}{(b_n - 1)(b_{n+1} - 1)}$, 求证: $d_1 + d_2 + d_3 + \dots + d_n < 2$;

(3) 记 $c_n = \begin{cases} \frac{1}{a_{2n-1}a_{2n+3}}, n=2k-1 \\ (2a_n-1) \cdot b_n, n=2k \end{cases}$ 其中 $k \in \mathbf{N}^*$, 求数列 $\{c_n\}$ 的前 $2n$ 项和 S_{2n} .

【解析】(1) $\because$ 数列 $\{a_n\}$ 是公差为 1 的等差数列, 且 $a_1 + a_2 = a_3$,

$$\therefore a_1 + (a_1 + 1) = a_1 + 2, \text{ 解得 } a_1 = 1,$$

$$\therefore a_n = a_1 + (n-1)d = n,$$

$\therefore$ 数列 $\{a_n\}$ 的通项公式为: $a_n = n, (n \in \mathbf{N}^*)$.

数列 $\{b_n\}$ 是等比数列, 且 $b_1 \cdot b_2 = b_3, a_4 = 4b_1 - b_2$,

设数列 $\{b_n\}$ 的公比为 q ,

$$\therefore \begin{cases} b_1 \cdot (b_1 q) = b_1 q^2 \\ 4 = 4b_1 - b_1 q \end{cases}, \text{ 解得 } b_1 = q = 2,$$

$$\therefore b_n = b_1 q^{n-1} = 2^n,$$

$\therefore$ 数列 $\{b_n\}$ 的通项公式为: $b_n = 2^n, (n \in \mathbf{N}^*)$.

(2) 由 (1) 知 $b_n = 2^n$,

$$\begin{aligned} \therefore d_n &= \frac{b_{n+1}}{(b_n - 1)(b_{n+1} - 1)} = \frac{2^{n+1}}{(2^n - 1)(2^{n+1} - 1)} = \frac{2 \times 2^n}{(2^n - 1)(2^{n+1} - 1)} = \frac{2 \times (2^{n+1} - 2^n)}{(2^n - 1)(2^{n+1} - 1)} \\ &= \frac{2 \times [(2^{n+1} - 1) - (2^n - 1)]}{(2^n - 1)(2^{n+1} - 1)} = 2 \left(\frac{1}{2^n - 1} - \frac{1}{2^{n+1} - 1} \right), \end{aligned}$$

$$\therefore d_1 + d_2 + d_3 + \dots + d_n = 2 \left(\frac{1}{2-1} - \frac{1}{2^2-1} \right) + 2 \left(\frac{1}{2^2-1} - \frac{1}{2^3-1} \right) + \dots + 2 \left(\frac{1}{2^n-1} - \frac{1}{2^{n+1}-1} \right)$$

$$= 2 \left(\frac{1}{2-1} - \frac{1}{2^{n+1}-1} \right) = 2 \left(1 - \frac{1}{2^{n+1}-1} \right),$$

$$\therefore n \in \mathbf{N}^*,$$

$$\therefore \frac{1}{2^{n+1}-1} > 0,$$

$$\therefore 1 - \frac{1}{2^{n+1}-1} < 1,$$

$$\therefore 2 \left(1 - \frac{1}{2^{n+1}-1} \right) < 2$$

$$\therefore d_1 + d_2 + d_3 + L + d_n < 2$$

$$(3) \text{ 由 (1) 可知 } a_n = n, a_{2n-1} = 2n-1, a_{2n+3} = 2n+3, b_n = 2^n,$$

$$\therefore c_n = \begin{cases} \frac{1}{(2n-1)(2n+3)}, n = 2k-1, (k \in \mathbf{N}^*), \\ (2n-1) \cdot 2^n, n = 2k \end{cases}$$

$$\therefore S_{2n} = (c_1 + c_2 + c_3 + c_4 + L + c_{2n-1} + c_{2n}) = (c_1 + c_3 + L + c_{2n-1}) + (c_2 + c_4 + L + c_{2n}),$$

$$\text{令 } A_n = c_1 + c_3 + L + c_{2n-1}, \quad B_n = c_2 + c_4 + L + c_{2n},$$

$$\begin{aligned} \therefore A_n &= \frac{1}{1 \times 5} + \frac{1}{5 \times 9} + L + \frac{1}{(4n-3)(4n+1)} \\ &= \frac{1}{4} \left(1 - \frac{1}{5} \right) + \frac{1}{4} \left(\frac{1}{5} - \frac{1}{9} \right) + L + \frac{1}{4} \left(\frac{1}{4n-7} - \frac{1}{4n-3} \right) + \frac{1}{4} \left(\frac{1}{4n-3} - \frac{1}{4n+1} \right) \\ &= \frac{1}{4} \left(1 - \frac{1}{4n+1} \right) = \frac{n}{4n+1}, \end{aligned}$$

$$B_n = 3 \times 2^2 + 7 \times 2^4 + L + (4n-5) \times 2^{2n-2} + (4n-1) \times 2^{2n},$$

$$\therefore 2^2 B_n = 3 \times 2^4 + 7 \times 2^6 + L + (4n-5) \times 2^{2n} + (4n-1) \times 2^{2n+2},$$

$$\therefore -3B_n = 3 \times 2^2 + 4 \times 2^4 + 4 \times 2^6 + L + 4 \times 2^{2n} - (4n-1) \times 2^{2n+2}$$

$$= -2^2 + [4 \times 2^2 + 4 \times 2^4 + 4 \times 2^6 + L + 4 \times 2^{2n}] - (4n-1) \times 2^{2n+2}$$

$$= -2^2 + 4 \times [2^2 + 2^4 + 2^6 + L + 2^{2n}] - (4n-1) \times 2^{2n+2}$$

$$= -4 + 4 \times \frac{4(4^n-1)}{4-1} - (4n-1) \cdot 2^{2n+2}$$

$$= \frac{1}{3} \times 4^{n+2} - (4n-1) \times 4^{n+1} - \frac{28}{3} = \frac{7-12n}{3} \times 4^{n+1} - \frac{28}{3},$$

$$\therefore B_n = \frac{12n-7}{9} \times 4^{n+1} + \frac{28}{9},$$

$$\therefore S_{2n} = A_n + B_n = \frac{n}{4n+1} + \frac{12n-7}{9} \times 4^{n+1} + \frac{28}{9},$$

$$\therefore \text{数列 } \{c_n\} \text{ 的前 } 2n \text{ 项和 } S_{2n} = A_n + B_n = \frac{n}{4n+1} + \frac{12n-7}{9} \times 4^{n+1} + \frac{28}{9}, (n \in \mathbf{N}^*).$$

二、等差数列与等比数列的证明

3. (2024 届贵州省贵阳市高三上学期 8 月考试) 设 S_n 为数列 $\{a_n\}$ 的前 n 项和. 已知 $4a_n - 3S_n = n$.

(1) 证明: 数列 $\left\{a_n + \frac{1}{3}\right\}$ 是等比数列;

(2) 设 $b_n = \log_2(3a_n + 1)$, 求数列 $\left\{\frac{1}{b_n b_{n+1}}\right\}$ 的前 n 项和 T_n .

【解析】(1) 证明: 已知 $4a_n - 3S_n = n$ ①,

当 $n \geq 2$ 时, $4a_{n-1} - 3S_{n-1} = n-1$ ②,

① - ② 得: $4a_n - 4a_{n-1} - 3a_n = 1$, 即 $a_n = 4a_{n-1} + 1$,

所以, $a_n + \frac{1}{3} = 4a_{n-1} + \frac{4}{3} = 4\left(a_{n-1} + \frac{1}{3}\right)$,

当 $n=1$ 时, 则 $a_1 = 4a_1 - 3S_1 = 1$, 则 $a_1 + \frac{1}{3} = \frac{4}{3}$,

所以, 数列 $\left\{a_n + \frac{1}{3}\right\}$ 是首项为 $\frac{4}{3}$, 公比为 4 的等比数列.

(2) 解: 由 (1) 可知, $a_n + \frac{1}{3} = \left(a_1 + \frac{1}{3}\right) \times 4^{n-1} = \frac{4^n}{3}$, 则 $a_n = \frac{4^n - 1}{3}$,

所以, $b_n = \log_2(3a_n + 1) = \log_2 4^n = 2n$,

所以, $\frac{1}{b_n b_{n+1}} = \frac{1}{2n(2n+2)} = \frac{1}{4n(n+1)} = \frac{1}{4} \left(\frac{1}{n} - \frac{1}{n+1} \right)$,

$$T_n = \frac{1}{b_1 b_2} + \frac{1}{b_2 b_3} + \cdots + \frac{1}{b_n b_{n+1}} = \frac{1}{4} \left(1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \cdots + \frac{1}{n} - \frac{1}{n+1} \right) = \frac{1}{4} \left(1 - \frac{1}{n+1} \right) = \frac{n}{4(n+1)}.$$

4. (2024 届湖南省常德市第一中学高三上学期第三次月考) 已知正项数列 $\{a_n\}$ 的前 n 项和为 S_n , $a_1 = 2$.

(1) 记 $c_n = \frac{a_{n+1}}{S_n \cdot S_{n+1}}$, 证明: 数列 $\{c_n\}$ 的前 n 项和 $T_n < \frac{1}{2}$;

(2) 若 $S_n = 2a_n + 14 - 2^{n+3}$ ($n \in \mathbf{N}^*$), 求证: 数列 $\left\{\frac{a_n}{2^n}\right\}$ 为等差数列, 并求 $\{a_n\}$ 的通项公式.

【解析】(1) $\because c_n = \frac{a_{n+1}}{S_n \cdot S_{n+1}} = \frac{1}{S_n} - \frac{1}{S_{n+1}}$,

$$\therefore T_n = \frac{1}{S_1} - \frac{1}{S_2} + \frac{1}{S_2} - \frac{1}{S_3} + \frac{1}{S_3} - \frac{1}{S_4} + \cdots + \frac{1}{S_{n-1}} - \frac{1}{S_n} + \frac{1}{S_n} - \frac{1}{S_{n+1}} = \frac{1}{a_1} - \frac{1}{S_{n+1}} = \frac{1}{2} - \frac{1}{S_{n+1}};$$

$\because$ 数列 $\{a_n\}$ 为正项数列, $\therefore S_{n+1} > 0$, $\therefore \frac{1}{2} - \frac{1}{S_{n+1}} < \frac{1}{2}$, 则 $T_n < \frac{1}{2}$.

(2) 当 $n \geq 2$ 且 $n \in \mathbf{N}^*$ 时, $S_{n-1} = 2a_{n-1} + 14 - 2^{n+2}$,

$$\therefore a_n = S_n - S_{n-1} = 2a_n + 14 - 2^{n+3} - 2a_{n-1} - 14 + 2^{n+2} = 2a_n - 2a_{n-1} - 2^{n+2},$$

$$\text{整理可得: } a_n - 2a_{n-1} = 2^{n+2}, \therefore \frac{a_n}{2^n} - \frac{a_{n-1}}{2^{n-1}} = 4,$$

经检验, 当 $n=1$ 时, $a_1 = S_1 = 2a_1 + 14 - 2^{1+3}$, 得 $a_1 = 2$, 满足条件,

又 $\frac{a_1}{2} = 1$, $\therefore$ 数列 $\left\{\frac{a_n}{2^n}\right\}$ 是以 1 为首项, 4 为公差的等差数列,

$$\therefore \frac{a_n}{2^n} = 1 + 4(n-1) = 4n-3, \therefore a_n = (4n-3) \cdot 2^n.$$

5. (2023 届陕西省西安市大明宫中学高三高考综合测试) 已知数列 $\{a_n\}$ 的各项均为正数, 且满足

$$2(a_1 + a_2 + a_3 + \cdots + a_n) = a_n(a_n + 1).$$

(1) 证明: 数列 $\{a_n\}$ 是等差数列;

(2) 求数列 $\left\{\frac{a_n}{2^n}\right\}$ 的前 n 项和 S_n .

【解析】(1) 由 $2(a_1 + a_2 + a_3 + \cdots + a_n) = a_n(a_n + 1)$,

$$\text{得 } 2(a_1 + a_2 + a_3 + \cdots + a_n + a_{n+1}) = a_{n+1}(a_{n+1} + 1),$$

$$\text{两式相减得 } 2a_{n+1} = a_{n+1}^2 + a_{n+1} - a_n^2 - a_n,$$

$$\text{即 } (a_{n+1} - a_n)(a_{n+1} + a_n) - (a_{n+1} + a_n) = 0,$$

$$\text{所以 } (a_{n+1} - a_n - 1)(a_{n+1} + a_n) = 0,$$

$$\text{又因 } a_n > 0, \text{ 所以 } a_{n+1} - a_n = 1,$$

$$\text{当 } n=1 \text{ 时, } 2a_1 = a_1(a_1 + 1), \text{ 解得 } a_1 = 1 \text{ (} a_1 = 0 \text{ 舍去),}$$

所以数列 $\{a_n\}$ 是以 1 为首项, 1 为公差的等差数列;

(2) 由 (1) 得 $a_n = n$,

$$\text{则 } \frac{a_n}{2^n} = \frac{n}{2^n},$$

$$\text{则 } S_n = \frac{1}{2} + \frac{2}{2^2} + \frac{3}{2^3} + \cdots + \frac{n}{2^n},$$

$$\frac{1}{2} S_n = \frac{1}{2^2} + \frac{2}{2^3} + \frac{3}{2^4} + \cdots + \frac{n}{2^{n+1}},$$

$$\text{两式相减得 } \frac{1}{2} S_n = \frac{1}{2} + \frac{1}{2^2} + \frac{2}{2^3} + \frac{3}{2^4} + \cdots + \frac{1}{2^n} - \frac{n}{2^{n+1}}$$

$$= \frac{\frac{1}{2} \left(1 - \frac{1}{2^n} \right)}{1 - \frac{1}{2}} - \frac{n}{2^{n+1}} = 1 - \frac{1}{2^n} - \frac{n}{2^{n+1}},$$

所以 $S_n = 2 - \frac{n+2}{2^n}$.

三、裂项求和

6. (2024 届四川省眉山市东坡区高三上学期开学考) 已知数列 $\{a_n\}$ 的前 n 项和为 S_n ,

$$\frac{a_1}{1} + \frac{a_2}{2} + \frac{a_3}{3} + \dots + \frac{a_n}{n} = n, \quad n \in \mathbf{N}^*.$$

(1) 求数列 $\{a_n\}$ 的通项公式;

(2) 若 a_1, a_k, S_{k+2} 成等比数列, $k \in \mathbf{N}^*$, 求 $\frac{1}{S_1} + \frac{1}{S_2} + \dots + \frac{1}{S_{k^2}}$ 的值.

【解析】(1) 数列 $\{a_n\}$ 的前 n 项和为 S_n , $\frac{a_1}{1} + \frac{a_2}{2} + \frac{a_3}{3} + \dots + \frac{a_n}{n} = n$ ①,

当 $n \geq 2$ 时, $\frac{a_1}{1} + \frac{a_2}{2} + \dots + \frac{a_{n-1}}{n-1} = (n-1)$ ②,

① - ② 得: $\frac{a_n}{n} = 1$, 所以 $a_n = n (n \geq 2)$,

又 $a_1 = 1$, 也满足上式, 故 $a_n = n$.

(2) 由于 $a_n = n$, 所以 $S_n = \frac{n(n+1)}{2}$, 故 $S_{k+2} = \frac{(k+2)(k+3)}{2}$,

由于 a_1, a_k, S_{k+2} 成等比数列, 所以 $k^2 = \frac{(k+2)(k+3)}{2}$,

解得 $k = 6$ 或 -1 (负值舍去),

$$\frac{1}{S_n} = \frac{2}{n(n+1)} = 2 \left(\frac{1}{n} - \frac{1}{n+1} \right),$$

所以 $\frac{1}{S_1} + \frac{1}{S_2} + \dots + \frac{1}{S_{k^2}} = \frac{1}{S_1} + \frac{1}{S_2} + \dots + \frac{1}{S_{36}} = 2 \times \left(1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \dots + \frac{1}{36} - \frac{1}{37} \right)$
 $= 2 \times \left(1 - \frac{1}{37} \right) = \frac{72}{37}.$

7. (2024 届安徽省皖东名校联盟体高三上学期 9 月第二次质量检测) 数列 $\{a_n\}$ 各项均为正数, $\{a_n\}$ 的前 n

项和记作 S_n , 已知 $S_1 = 1$, $a_n^2 - a_n - 2S_{n-1} = 0, (n \geq 2)$.

(1) 求 $\{a_n\}$ 的通项公式;

(2) 设 $b_n = \tan(a_n) \cdot \tan(a_{n+1})$, 求数列 $\{b_n\}$ 的前 2023 项和.

【解析】(1) 当 $n \geq 2$ 时, 有 $\begin{cases} a_{n+1}^2 - a_{n+1} = 2S_n \\ a_n^2 - a_n = 2S_{n-1} \end{cases}$ 相减得 $a_{n+1}^2 - a_n^2 + a_n - a_{n+1} = 2a_n$, 即 $(a_{n+1} - a_n - 1)(a_n + a_{n+1}) = 0$,

$\{a_n\}$ 各项均为正数,

所以 $a_{n+1} = a_n + 1 (n \geq 2)$,

又当 $n = 2$ 时, $a_2^2 - a_2 - 2S_1 = a_2^2 - a_2 - 2 = 0$,

解得 $a_2 = 2$ 或 $a_2 = -1$ (舍),

所以对任意正整数 n , 均有 $a_{n+1} = a_n + 1$,

故 $\{a_n\}$ 是以首项为 1, 公差为 1 的等差数列,

所以 $a_n = n (n \in \mathbf{N}^*)$.

(2) 由于 $\tan 1 = \tan[(n+1) - n] = \frac{\tan(n+1) - \tan n}{1 + \tan(n+1)\tan n}$,

故 $\tan(n+1)\tan n = \frac{\tan(n+1) - \tan n}{\tan 1} - 1$,

由 (1) 得 $b_n = \tan(a_n)\tan(a_{n+1}) = \tan n \cdot \tan(n+1)$,

记 $\{b_n\}$ 前 n 项和为 T_n , 则 $T_n = b_1 + b_2 + b_3 + \dots + b_n = \frac{1}{\tan 1} [\tan(n+1) - \tan n + \tan n - \tan(n-1) + \dots + \tan 2 - \tan 1] - n$

$$= \frac{1}{\tan 1} [\tan(n+1) - \tan 1] - n$$

$$= \frac{\tan(n+1)}{\tan 1} - n - 1,$$

所以 $T_{2023} = \frac{\tan 2024}{\tan 1} - 2024$.

8. (2024 届黑龙江省哈尔滨工业大学附属中学校高三上学期 9 月月考) 已知数列 $\{a_n\}$, S_n 是数列 $\{a_n\}$ 的前

n 项和, 满足 $S_n = n^2$; 数列 $\{b_n\}$ 是正项的等比数列, T_n 是数列 $\{b_n\}$ 的前 n 项和, 满足 $b_1 = 1$, $T_3 = 7$

$(n \in \mathbf{N}^*)$.

(1) 求数列 $\{a_n\}$ 和 $\{b_n\}$ 的通项公式;

(2) 记 $c_n = \begin{cases} \frac{6n+13}{a_n a_{n+2} 2^{n+1}}, & n \text{ 为奇数} \\ \log_2 b_{n+1}, & n \text{ 为偶数} \end{cases}$, 数列 $\{c_n\}$ 的前 $2n$ 项和为 K_{2n} , 若不等式 $(-1)^n \lambda - \frac{1}{(4n+1)4^n} < K_{2n}$ 对一切

$n \in \mathbf{N}^*$ 恒成立, 求 λ 的取值范围.

【解析】(1) 依题意 $S_n = n^2$;

当 $n \geq 2$ 时, $a_n = S_n - S_{n-1} = n^2 - (n-1)^2 = 2n-1$; 当 $n=1$ 时, $a_1 = S_1 = 1$ 适合上式,

所以数列 $\{a_n\}$ 的通项公式 $a_n = 2n-1$.

又因为 $b_1=1, T_3=1+q+q^2=7$, 数列 $\{b_n\}$ 为等比数列,

所以 $q^2+q-6=0$, 解得 $q=2$ 或 $q=-3$ (舍去), 所以 $b_n=2^{n-1}$;

(2) 由题意可知, $S_n=n^2$, $b_{n+1}=2^n$;

由已知 $c_n = \begin{cases} \frac{6n+13}{(2n-1)(2n+3)2^{n+1}}, & n \text{ 为奇数} \\ n, & n \text{ 为偶数} \end{cases}$

设 $\{c_n\}$ 的前 $2n$ 项和中, 奇数项的和为 P_n , 偶数项的和为 Q_n ,

所以 $P_n = c_1 + c_3 + c_5 + \dots + c_{2n-1}$, $Q_n = c_2 + c_4 + c_6 + \dots + c_{2n}$,

当 n 为奇数时, $c_n = \frac{6n+13}{(2n-1)(2n+3)2^{n+1}} = \frac{1}{(2n-1)2^{n-1}} - \frac{1}{(2n+3)2^{n+1}}$,

所以, $P_n = 1 - \frac{1}{5 \cdot 2^2} + \frac{1}{5 \cdot 2^2} - \frac{1}{9 \cdot 2^4} + \dots - \frac{1}{(4n+1)2^{2n}} = 1 - \frac{1}{(4n+1)4^n}$

当 n 为偶数时, $c_n = n$, 所以

$Q_n = c_2 + c_4 + c_6 + \dots + c_{2n} = 2 + 4 + 6 + \dots + 2n = \frac{(2+2n)n}{2} = n(n+1)$,

由 $(-1)^n \lambda - \frac{1}{(4n+1)4^n} < K_{2n}$, 得 $(-1)^n \lambda - \frac{1}{(4n+1)4^n} < 1 - \frac{1}{(4n+1)4^n} + n(n+1)$, 即 $(-1)^n \lambda < n^2 + n + 1$,

当 n 为偶数时, $\lambda < n^2 + n + 1$ 对一切偶数成立, 当 $n=2$ 时, $n^2 + n + 1 = 7$ 为最小值, 所以 $\lambda < 7$,

当 n 为奇数时, $-\lambda < n^2 + n + 1$ 对一切奇数成立, 当 $n=1$ 时, $-(n^2 + n + 1) = -3$ 为最大值,

所以此时 $\lambda > -3$,

故对一切 $n \in \mathbf{N}^*$ 恒成立, 则 $-3 < \lambda < 7$.

9. (2024 届湖北省黄冈市高三上学期 9 月调研) 设等差数列 $\{a_n\}$ 前 n 项和 S_n , $a_1=1$, 满足

$2S_{n+1} = n(a_n + 5) + 2$, $n \in \mathbf{N}^*$.

(1) 求数列 $\{a_n\}$ 的通项公式;

(2) 记 $b_n = \frac{n+1}{S_n S_{n+2}}$, 设数列 $\{b_n\}$ 的前 n 项和为 T_n , 求证 $T_n < \frac{5}{16}$.

【解析】(1) 依题意有 $2(a_1 + a_2) = a_1 + 5 + 2$,

$$\text{Q } a_1 = 1, \therefore a_2 = 3,$$

又 $\{a_n\}$ 为等差数列, 设公差为 d ,

$$\therefore d = a_2 - a_1 = 2, \therefore a_n = 1 + 2(n-1) = 2n-1.$$

$$(2) \text{ 由 (1) 可得 } S_n = \frac{(1+2n-1)n}{2} = n^2,$$

$$\therefore b_n = \frac{n+1}{n^2(n+2)^2} = \frac{1}{4} \left(\frac{1}{n^2} - \frac{1}{(n+2)^2} \right)$$

$$b_1 = \frac{1}{4} \left(\frac{1}{1^2} - \frac{1}{3^2} \right), b_2 = \frac{1}{4} \left(\frac{1}{2^2} - \frac{1}{4^2} \right), b_3 = \frac{1}{4} \left(\frac{1}{3^2} - \frac{1}{5^2} \right), \dots, b_{n-1} = \frac{1}{4} \left(\frac{1}{(n-1)^2} - \frac{1}{(n+1)^2} \right),$$

$$\therefore T_n = \frac{1}{4} \left(1 + \frac{1}{4} - \frac{1}{(n+1)^2} - \frac{1}{(n+2)^2} \right) < \frac{1}{4} \times \frac{5}{4} = \frac{5}{16}.$$

四、错位相减法求和

10. (2024 届湖南省邵阳市邵东市高三上学期第二次月考) 已知数列 $\{a_n\}$ 满足

$$a_1 = 2, a_1 + \frac{a_2}{2} + \frac{a_3}{3} + \dots + \frac{a_n}{n} = a_{n+1} - 2, \text{ 数列 } \{b_n\} \text{ 满足 } b_1 = \frac{1}{8}, b_2 = \frac{1}{32}, b_{n+1}^2 = b_n \cdot b_{n+2}.$$

(1) 求数列 $\{a_n\}$ 和 $\{b_n\}$ 的通项公式;

(2) 设 $c_n = a_n \cdot b_n$, 求数列 $\{c_n\}$ 的前 n 项和 T_n .

$$\text{【解析】(1) 由题意知 } a_1 + \frac{a_2}{2} + \frac{a_3}{3} + \dots + \frac{a_n}{n} = a_{n+1} - 2,$$

$$\text{则当 } n \geq 2 \text{ 时, } a_1 + \frac{a_2}{2} + \frac{a_3}{3} + \dots + \frac{a_{n-1}}{n-1} = a_n - 2,$$

$$\text{故两式相减得 } \frac{a_n}{n} = a_{n+1} - a_n, \text{ 即 } \frac{a_{n+1}}{a_n} = \frac{n+1}{n},$$

$$\text{又当 } n=1 \text{ 时, } a_1 = a_2 - 2, a_1 = 2, \text{ 故 } a_2 = 4,$$

$$\text{即 } \frac{a_2}{a_1} = 2 \text{ 也适合 } \frac{a_{n+1}}{a_n} = \frac{n+1}{n};$$

$$\text{所以当 } n \geq 2 \text{ 时, } a_n = a_1 \cdot \frac{a_2}{a_1} \cdot \frac{a_3}{a_2} \cdot \frac{a_4}{a_3} \cdot \dots \cdot \frac{a_n}{a_{n-1}} = 2 \times \frac{2}{1} \times \frac{3}{2} \times \frac{4}{3} \times \dots \times \frac{n}{n-1} = 2n,$$

$$\text{即 } a_n = 2n, a_1 = 2 \text{ 也适合, 故 } a_n = 2n;$$

$$\text{又数列 } \{b_n\} \text{ 满足 } b_1 = \frac{1}{8}, b_2 = \frac{1}{32}, b_{n+1}^2 = b_n \cdot b_{n+2},$$

$$\text{则 } \{b_n\} \text{ 为等比数列, 设公比为 } q, \text{ 则 } q = \frac{b_2}{b_1} = \frac{1}{4},$$

故 $b_n = b_1 q^{n-1} = \frac{1}{8} \times (\frac{1}{4})^{n-1} = (\frac{1}{2})^{2n+1}$ ，即 $b_n = \frac{1}{2^{2n+1}}$ ；

(2) 由 (1) 可得 $c_n = a_n \cdot b_n = 2n \times \frac{1}{2^{2n+1}} = n \times \frac{1}{2^{2n}}$ ，

故 $T_n = \frac{1}{4} + 2 \times \frac{1}{4^2} + 3 \times \frac{1}{4^3} + \dots + n \times \frac{1}{4^n}$ ，

则 $\frac{1}{4}T_n = \frac{1}{4^2} + 2 \times \frac{1}{4^3} + 3 \times \frac{1}{4^4} + \dots + (n-1) \times \frac{1}{4^n} + n \times \frac{1}{4^{n+1}}$ ，

故 $\frac{3}{4}T_n = \frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} + \frac{1}{4^4} + \dots + \frac{1}{4^n} - n \times \frac{1}{4^{n+1}}$

$$= \frac{\frac{1}{4}(1 - \frac{1}{4^n})}{1 - \frac{1}{4}} - n \times \frac{1}{4^{n+1}} = \frac{1}{3} - \frac{3n+4}{12} \cdot \frac{1}{4^n}，$$

故 $T_n = \frac{4}{9} - \frac{3n+4}{9} \cdot \frac{1}{4^n}$ 。

11. (2024 届广东省南粤名校高三上学期 9 月联考) 已知数列 $\{a_n\}$ 的首项 $a_1 = 3$ ，其前 n 项和为 S_n ，且

$$S_{n+1} = 3S_n + 2n + 3, \quad (n \in \mathbb{N}^*)$$

(1) 求数列 $\{a_n\}$ 的通项公式 a_n ；

(2) 设 $b_n = na_n$ ，求数列 $\{b_n\}$ 的前 n 项和 T_n 。

【解析】(1) 由已知 $S_{n+1} = 3S_n + 2n + 3$ ，

$\therefore n \geq 2$ 时， $S_n = 3S_{n-1} + 2n + 1$ ，

两式相减，得 $S_{n+1} - S_n = 3(S_n - S_{n-1}) + 2$ ，

即 $a_{n+1} = 3a_n + 2$ ，从而 $a_{n+1} + 1 = 3(a_n + 1)$ ，

又当 $n=1$ 时， $S_2 = 3S_1 + 5$ ，

$\therefore a_1 + a_2 = 3a_1 + 5$ 又 $a_1 = 3$ ， $\therefore a_2 = 11$ ，

从而 $a_2 + 1 = 3(a_1 + 1)$ 。

故总有 $a_{n+1} + 1 = 3(a_n + 1)$ ， $n \in \mathbb{N}^*$ 。

又 $\because a_1 = 3$ ， $\therefore a_n + 1 \neq 0$ ，从而 $\frac{a_{n+1} + 1}{a_n + 1} = 3$ 。

即 $\{a_n + 1\}$ 是以 $a_1 + 1 = 4$ 为首项，公比为 3 的等比数列。

$\therefore a_n + 1 = 4 \cdot 3^{n-1}$ ，

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