第四章

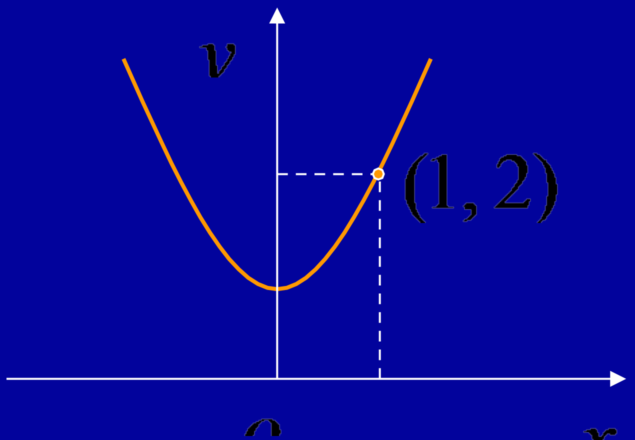
例题及习题

例1. 设曲线经过点(1 , 2) , 且其上任一点处切线斜率等于该点横坐标两倍, 求此曲线方程.

解:

O $y' = 2x$

$$\therefore \quad y = \int 2x\mathrm{d}x = x^2 + C$$



所求曲线过点 (1 , 2) , 故有

$$2 = 1^2 + C$$

$$\therefore \quad C = 1$$

所以所求曲线为 $y = x^2 + 1$

例2. 求 $\int \frac{\mathrm{d}x}{x\sqrt[3]{x}}$.

解: 原式 $= \int x^{-\frac{4}{3}}\mathrm{d}x = \frac{x^{-\frac{4}{3}+1}}{-\frac{4}{3}+1} + C$

$= -3x^{-\frac{1}{3}} + C$

例3. 求 $\int \sin\frac{x}{2}\cos\frac{x}{2}\mathrm{d}x$.

解: 原式$= \int \frac{1}{2}\sin x\,\mathrm{d}x = -\frac{1}{2}\cos x + C$

例4. 求 $\int 2^x(e^x-5)\mathrm{d}x$.

解: 原式 $= \int [(2e)^x - 5\cdot 2^x)\mathrm{d}x$

$$= \frac{(2e)^x}{\ln(2e)} - 5\frac{2^x}{\ln 2} + C$$

$$= 2^x\left[\frac{e^x}{\ln 2+1} - \frac{5}{\ln 2}\right] + C$$

例5. 求 $\int \tan^2 x\mathrm{d}x$.

解: 原式 $= \int (\sec^2 x - 1)\mathrm{d}x$

$$= \int \sec^2 x\mathrm{d}x - \int \mathrm{d}x = \tan x - x + C$$

例6. 求 $\int \frac{1+x+x^2}{x(1+x^2)}\mathrm{d}x$.

解: 原式 $= \int \frac{x+(1+x^2)}{x(1+x^2)}\mathrm{d}x$

$$= \int \frac{1}{1+x^2}\mathrm{d}x + \int \frac{1}{x}\mathrm{d}x$$

$$= \arctan x + \ln|x| + C$$

例7. 求 $\int \frac{x^4}{1+x^2}\,\mathrm{d}x$.

解: 原式 $= \int \frac{(x^4-1)+1}{1+x^2}\,\mathrm{d}x$

$$= \int \frac{(x^2-1)(x^2+1)+1}{1+x^2}\,\mathrm{d}x$$

$$= \int (x^2-1)\,\mathrm{d}x + \int \frac{\mathrm{d}x}{1+x^2}$$

$$= \frac{1}{3}x^3 - x + \arctan x + C$$

思索与练习

1. 证实 $\frac{1}{2}e^{2x}, e^x \operatorname{sh} x, e^x \operatorname{ch} x$ 都是 $\underline{\dfrac{e^x}{\operatorname{ch} x - \operatorname{sh} x}}$ 的原函数

提醒: $\operatorname{ch} x = \dfrac{e^x + e^{-x}}{2}$, $\operatorname{sh} x = \dfrac{e^x - e^{-x}}{2}$

2. 若 e^{-x} 是 $f(x)$ 的原函数则

$$\int x^2 f(\ln x)\,\mathrm{d}x = \underline{-\frac{1}{2}x^2 + C}$$

提醒: $f(x) = (e^{-x})' = -e^{-x}$

$$f(\ln x) = -e^{-\ln x} = -\frac{1}{x}$$

3. 若 $f(x)$ 导函数为 $\sin x$, 则 $f(x)$ 一个原函数是 (***B***).

$(A)\quad 1+\sin x;$ $(B)\quad 1-\sin x;$

$(C)\quad 1+\cos x;$ $(D)\quad 1-\cos x.$

提醒: 已知 $f'(x)=\sin x$

求 $(\ ?\ )'=f(x)$

即 $(\ ?\ )''=\sin x$

4. 求以下积分:

(1) $\int \frac{\mathrm{d}x}{x^2(1+x^2)}$;　　(2) $\int \frac{\mathrm{d}x}{\sin^2 x \cos^2 x}$.

提醒:

(1) $$\frac{1}{x^2(1+x^2)} = \frac{(1+x^2)-x^2}{x^2(1+x^2)} = \frac{1}{x^2} - \frac{1}{1+x^2}$$

(2) $$\frac{1}{\sin^2 x \cos^2 x} = \frac{\sin^2 x + \cos^2 x}{\sin^2 x \cos^2 x}$$

$$= \sec^2 x + \csc^2 x$$

5. 求不定积分 $\int \frac{e^{3x}+1}{e^x+1}\mathrm{d}x$.

解:

$$\int \frac{e^{3x}+1}{e^x+1}\mathrm{d}x$$

$$=\int \frac{(e^x+1)(e^{2x}-e^x+1)}{e^x+1}\mathrm{d}x$$

$$=\int (e^{2x}-e^x+1)\mathrm{d}x$$

$$=\frac{1}{2}e^{2x}-e^x+x+C$$

例1. 求 $\int (ax+b)^m \mathrm{d}x \quad (m \neq -1)$.

解: 令 $u = ax+b$，则 $\mathrm{d}u = a\mathrm{d}x$，故

$$\text{原式} = \int u^m \frac{1}{a} \mathrm{d}u = \frac{1}{a} \cdot \frac{1}{m+1} u^{m+1} + C$$

$$= \frac{1}{a(m+1)} (ax+b)^{m+1} + C$$

注: 当 $m = -1$ 时

$$\int \frac{\mathrm{d}x}{ax+b} = \frac{1}{a} \int \frac{\mathrm{d}(ax+b)}{ax+b} = \frac{1}{a} \ln|ax+b| + C$$

例2. 求 $\int \frac{\mathrm{d}x}{a^2+x^2}$.

想到公式

$$\int \frac{\mathrm{d}u}{1+u^2} = \arctan u + C$$

解: $\int \frac{\mathrm{d}x}{a^2+x^2} = \frac{1}{a^2}\int \frac{\mathrm{d}x}{1+(\frac{x}{a})^2}$

令 $u=\frac{x}{a}$, 则 $\mathrm{d}u = \frac{1}{a}\mathrm{d}x$

$$\frac{1}{a}\int \frac{\mathrm{d}u}{1+u^2} = \frac{1}{a}\arctan u + C$$

$$= \frac{1}{a}\arctan(\frac{x}{a}) + C$$

例3. 求 $\int \frac{\mathrm{d}x}{\sqrt{a^2-x^2}}\ (a>0)$.

解: $\int \frac{\mathrm{d}x}{\sqrt{a^2-x^2}} = \int \frac{\mathrm{d}x}{a\sqrt{1-(\frac{x}{a})^2}} = \int \frac{\mathrm{d}(\frac{x}{a})}{\sqrt{1-(\frac{x}{a})^2}}$

$$= \arcsin\frac{x}{a}+C$$

想到 $\int \frac{\mathrm{d}u}{\sqrt{1-u^2}} = \arcsin u + C$

$$\int f[\varphi(x)]\varphi'(x)\mathrm{d}x = \int f(\varphi(x))\mathrm{d}\varphi(x)$$ (直接凑微分)

例4. 求 $\int \tan x \mathrm{d}x$.

解: $\int \tan x \mathrm{d}x = \int \frac{\sin x}{\cos x}\mathrm{d}x = -\int \frac{\mathrm{d}\cos x}{\cos x}$

$= -\ln\left|\cos x\right| + C$

类似

$\int \cot x \mathrm{d}x = \int \frac{\cos x\,\mathrm{d}x}{\sin x} = \int \frac{\mathrm{d}\sin x}{\sin x}$

$= \ln\left|\sin x\right| + C$

例5. 求 $\int \frac{\mathrm{d}x}{x^2 - a^2}$.

解:

$$Q \quad \frac{1}{x^2 - a^2} = \frac{1}{(x-a)(x+a)} = \frac{1}{2a}\left(\frac{1}{x-a} - \frac{1}{x+a}\right)$$

$$\therefore \text{原式} = \frac{1}{2a}\left[\int \frac{\mathrm{d}x}{x-a} - \int \frac{\mathrm{d}x}{x+a}\right]$$

$$= \frac{1}{2a}\left[\int \frac{\mathrm{d}(x-a)}{x-a} - \int \frac{\mathrm{d}(x+a)}{x+a}\right]$$

$$= \frac{1}{2a}\left[\ln|x-a| - \ln|x+a|\right] + C = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + C$$

惯用几个配元形式:

$$(1)\ \int f(ax+b)\mathrm{d}x = \frac{1}{a}\int f(ax+b)\ \mathrm{d}(ax+b)$$

$$(2)\ \int f(x^n)x^{n-1}\,\mathrm{d}x = \frac{1}{n}\int f(x^n)\ \mathrm{d}x^n$$

$$(3)\ \int f(x^n)\frac{1}{x}\mathrm{d}x = \frac{1}{n}\int f(x^n)\ \frac{1}{x^n}\ \mathrm{d}x^n$$

万能凑幂法

$$(4)\ \int f(\sin x)\cos x\mathrm{d}x = \int f(\sin x)\,\mathrm{d}\sin x$$

$$(5)\ \int f(\cos x)\sin x\mathrm{d}x = \int f(\cos x)\ \mathrm{d}\cos x$$

$$(6)\ \int f(\tan x)\sec^2 x\mathrm{d}x = \int f(\tan x)\ \mathrm{d}\tan x$$

$$(7)\ \int f(e^x)e^x\mathrm{d}x = \int f(e^x)\ \mathrm{d}e^x$$

$$(8)\ \int f(\ln x)\frac{1}{x}\mathrm{d}x = \int f(\ln x)\ \mathrm{d}\ln x$$

例6. 求 $\int \frac{\mathrm{d}x}{x(1+2\ln x)}$.

解: 原式 $= \int \frac{\mathrm{d}\ln x}{1+2\ln x} = \frac{1}{2}\int \frac{\mathrm{d}(1+2\ln x)}{1+2\ln x}$

$$= \frac{1}{2}\ln\left|1+2\ln x\right| + C$$

例7. 求 $\int \frac{e^{3\sqrt{x}}}{\sqrt{x}}\mathrm{d}x$.

解: 原式 $= 2\int e^{3\sqrt{x}}\mathrm{d}\sqrt{x} = \frac{2}{3}\int e^{3\sqrt{x}}\mathrm{d}(3\sqrt{x})$

$= \frac{2}{3}e^{3\sqrt{x}} + C$

例8. 求 $\int \sec^6 x\mathrm{d}x$.

解: 原式 $= \int (\tan^2 x + 1)^2 \,\mathrm{d}\tan x$

$= \int (\tan^4 x + 2\tan^2 x + 1)\,\mathrm{d}\tan x$

$= \frac{1}{5}\tan^5 x + \frac{2}{3}\tan^3 x + \tan x + C$

例9. 求 $\int \frac{\mathrm{d}x}{1+e^x}$.

解法1

$$\int \frac{\mathrm{d}x}{1+e^x} = \int \frac{(1+e^x)-e^x}{1+e^x}\,\mathrm{d}x = \int \mathrm{d}x - \int \frac{\mathrm{d}(1+e^x)}{1+e^x}$$

$$= x - \ln(1+e^x) + C$$

解法2

$$\int \frac{\mathrm{d}x}{1+e^x} = \int \frac{e^{-x}}{1+e^{-x}}\,\mathrm{d}x = -\int \frac{\mathrm{d}(1+e^{-x})}{1+e^{-x}}$$

$$= -\ln(1+e^{-x}) + C$$

$-\ln(1+e^{-x}) = -\ln[e^{-x}(e^x+1)]$ 两法结果一样

例10. 求 $\int \sec x \mathrm{d}x$.

解法1

$$\int \sec x \mathrm{d}x = \int \frac{\cos x}{\cos^2 x}\mathrm{d}x = \int \frac{\mathrm{d}\sin x}{1-\sin^2 x}$$

$$\frac{1}{2}\int\left[\frac{1}{1+\sin x}+\frac{1}{1-\sin x}\right]\mathrm{d}\sin x$$

$$= \frac{1}{2}\left[\ln\left|1+\sin x\right| - \ln\left|1-\sin x\right|\right] + C$$

$$= \frac{1}{2}\ln\left|\frac{1+\sin x}{1-\sin x}\right| + C$$

解法 2 $\int \sec x \mathrm{d}x = \int \frac{\sec x(\sec x + \tan x)}{\sec x + \tan x}\mathrm{d}x$

$$= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x}\mathrm{d}x$$

$$= \int \frac{\mathrm{d}(\sec x + \tan x)}{\sec x + \tan x}$$

$$= \ln\left| \sec x + \tan x \right| + C$$

一样可证

$$\int \csc x \mathrm{d}x = \ln\left| \csc x - \cot x \right| + C$$

或 $\int \csc x \mathrm{d}x = \ln\left| \tan\frac{x}{2} \right| + C$

例11. 求 $\int \frac{x^3}{(x^2+a^2)^{3/2}}\mathrm{d}x$.

解: 原式 $= \frac{1}{2}\int \frac{x^2\,\mathrm{d}x^2}{(x^2+a^2)^{3/2}} = \frac{1}{2}\int \frac{(x^2+a^2)-a^2}{(x^2+a^2)^{3/2}}\mathrm{d}x^2$

$$= \frac{1}{2}\int (x^2+a^2)^{-1/2}\,\mathrm{d}(x^2+a^2)$$

$$-\frac{a^2}{2}\int (x^2+a^2)^{-3/2}\,\mathrm{d}(x^2+a^2)$$

$$= \sqrt{x^2+a^2} + \frac{a^2}{\sqrt{x^2+a^2}} + C$$

例12. 求 $\int \cos^4 x\,\mathrm{d}x$.

解: $\because \cos^4 x = (\cos^2 x)^2 = (\frac{1+\cos 2x}{2})^2$

$= \frac{1}{4}(1 + 2\cos 2x + \cos^2 2x)$

$= \frac{1}{4}(1 + 2\cos 2x + \frac{1+\cos 4x}{2})$

$= \frac{1}{4}(\frac{3}{2} + 2\cos 2x + \frac{1}{2}\cos 4x)$

$\therefore \int \cos^4 x\,\mathrm{d}x = \frac{1}{4}\int (\frac{3}{2} + 2\cos 2x + \frac{1}{2}\cos 4x)\,\mathrm{d}x$

$= \frac{1}{4}[\frac{3}{2}\int \mathrm{d}x + \int \cos 2x\,\mathrm{d}(2x) + \frac{1}{8}\int \cos 4x\,\mathrm{d}(4x)]$

$= \frac{3}{8}x + \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + C$

例13. 求 $\int \sin^2 x \cos^2 3x \,\mathrm{d}x$.

解: $Q\ \sin^2 x \cos^2 3x = [\frac{1}{2}(\sin 4x - \sin 2x)]^2$

$$= \frac{1}{4}\sin^2 4x - \frac{1}{4}\cdot 2\sin 4x \sin 2x + \frac{1}{4}\sin^2 2x$$

$$= \frac{1}{8}(1-\cos 8x) - \sin^2 2x \cos 2x + \frac{1}{8}(1-\cos 4x)$$

$\therefore$ 原式 $= \frac{1}{4}\int \mathrm{d}x - \frac{1}{64}\int \cos 8x \,\mathrm{d}(8x)$

$$-\frac{1}{2}\int \sin^2 2x \,\mathrm{d}(\sin 2x) - \frac{1}{32}\int \cos 4x \,\mathrm{d}(4x)$$

$$= \frac{1}{4}x - \frac{1}{64}\sin 8x - \frac{1}{6}\sin^3 2x - \frac{1}{32}\sin 4x + C$$

例14. 求 $\int \frac{x+1}{x(1+xe^x)}\mathrm{d}x$.

解: 原式= $\int \frac{(x+1)\,e^x}{xe^x(1+xe^x)}\mathrm{d}x = \int (\frac{1}{xe^x} - \frac{1}{1+xe^x})\mathrm{d}(xe^x)$

$$= \ln\left|xe^x\right| - \ln\left|1+xe^x\right| + C$$

$$= x + \ln|x| - \ln\left|1+xe^x\right| + C$$

分析: $\frac{1}{xe^x(1+xe^x)} = \frac{1+xe^x - xe^x}{xe^x(1+xe^x)} = \frac{1}{xe^x} - \frac{1}{1+xe^x}$

$$(x+1)e^x\,\mathrm{d}x = xe^x\,\mathrm{d}x + e^x\,\mathrm{d}x = \mathrm{d}(xe^x)$$

例15. 求 $\int\left[\frac{f(x)}{f'(x)}-\frac{f''(x)f^2(x)}{f'^3(x)}\right]\mathrm{d}x.$

解：原式 $=\int\frac{f(x)}{f'(x)}\left[1-\frac{f''(x)f(x)}{f'^2(x)}\right]\mathrm{d}x$

$$=\int\frac{f(x)}{f'(x)}\cdot\frac{f'^2(x)-f''(x)f(x)}{f'^2(x)}\mathrm{d}x$$

$$=\int\frac{f(x)}{f'(x)}\,\mathrm{d}\left(\frac{f(x)}{f'(x)}\right)$$

$$=\frac{1}{2}\left[\frac{f(x)}{f'(x)}\right]^2+C$$

小结 惯用简化技巧:

(1) 分项积分: 利用积化和差; 分式分项;

$$1=\sin^2 x+\cos^2 x \text{ 等恒等式}$$

(2) 降低幂次: 利用倍角公式 , 如

$$\cos^2 x=\frac{1}{2}(1+\cos 2x)\,;\qquad \sin^2 x=\frac{1}{2}(1-\cos 2x)\,;$$

万能凑幂法
$$\begin{cases}\int f(x^n)x^{n-1}\,\mathrm{d}x=\frac{1}{n}\int f(x^n)\,\mathrm{d}\,x^n\\ \int f(x^n)\frac{1}{x}\,\mathrm{d}x=\frac{1}{n}\int f(x^n)\,\frac{1}{x^n}\,\mathrm{d}\,x^n\end{cases}$$

(3) 统一函数: 利用三角公式 ; 配元方法

(4) 巧妙换元或配元

思索与练习　1. 以下各题求积方法有何不一样?

(1) $\int \frac{dx}{4+x} = \int \frac{d(4+x)}{4+x}$　　(2) $\int \frac{dx}{4+x^2} = \frac{1}{2}\int \frac{d(\frac{x}{2})}{1+(\frac{x}{2})^2}$

(3) $\int \frac{x}{4+x^2}dx = \frac{1}{2}\int \frac{d(4+x^2)}{4+x^2}$

(4) $\int \frac{x^2}{4+x^2}dx = \int [1-\frac{4}{4+x^2}]dx$

(5) $\int \frac{dx}{4-x^2} = \frac{1}{4}\int [\frac{1}{2-x}+\frac{1}{2+x}]dx$

(6) $\int \frac{dx}{\sqrt{4x-x^2}} = \int \frac{d(x-2)}{\sqrt{4-(x-2)^2}}$

2. 求 $\int \frac{\mathrm{d}x}{x(x^{10}+1)}$.

提醒:

法1 $\int \frac{\mathrm{d}x}{x(x^{10}+1)} = \int \frac{(x^{10}+1)-x^{10}}{x(x^{10}+1)}\mathrm{d}x$

法2 $\int \frac{\mathrm{d}x}{x(x^{10}+1)} = \frac{1}{10}\int \frac{\mathrm{d}\,x^{10}}{x^{10}(x^{10}+1)}$

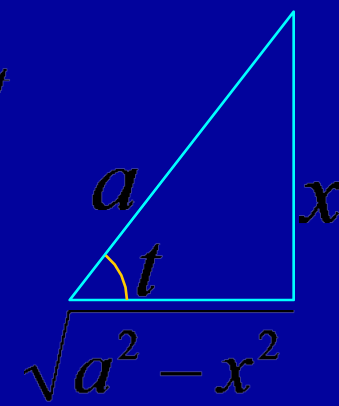
法3 $\int \frac{\mathrm{d}x}{x(x^{10}+1)} = \int \frac{\mathrm{d}x}{x^{11}(1+x^{-10})} = \frac{-1}{10}\int \frac{\mathrm{d}\,x^{-10}}{1+x^{-10}}$

例16. 求 $\int \sqrt{a^2 - x^2}\, \mathrm{d}x \ \ (a > 0)$.

解: 令 $x = a\sin t,\ t \in (-\frac{\pi}{2}, \frac{\pi}{2})$, 则

$$\sqrt{a^2 - x^2} = \sqrt{a^2 - a^2\sin^2 t} = a\cos t$$

$$\mathrm{d}x = a\cos t\, \mathrm{d}t$$



$\therefore$ 原式 $= \int a\cos t \cdot a\cos t\, \mathrm{d}t = a^2 \int \cos^2 t\, \mathrm{d}t$

$$= a^2 \left(\frac{t}{2} + \frac{\sin 2t}{4} \right) + C$$

$$\sin 2t = 2\sin t\cos t = 2 \cdot \frac{x}{a} \cdot \frac{\sqrt{a^2 - x^2}}{a}$$

$$= \frac{a^2}{2}\arcsin\frac{x}{a} + \frac{1}{2}x\sqrt{a^2 - x^2} + C$$

(切记变量还原)

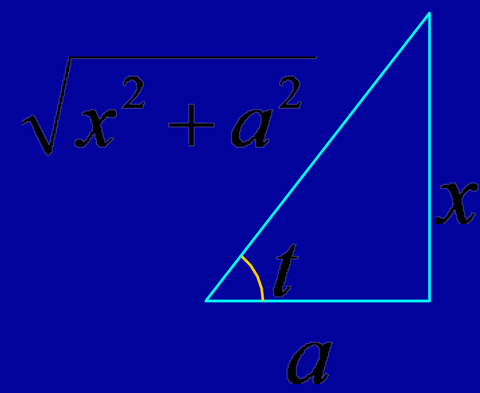
例17. 求 $\int \frac{\mathrm{d}x}{\sqrt{x^2+a^2}} \quad (a>0)$.

解: 令 $x = a\tan t,\ t\in(-\frac{\pi}{2},\frac{\pi}{2})$, 则

$$\sqrt{x^2+a^2} = \sqrt{a^2\tan^2 t + a^2} = a\sec t$$

$$\mathrm{d}x = a\sec^2 t\,\mathrm{d}t$$

$$\therefore\ 原式 = \int \frac{a\sec^2 t}{a\sec t}\mathrm{d}t = \int \sec t\,\mathrm{d}t$$



$$= \ln\left|\sec t + \tan t\right| + C_1$$

$$= \ln\left[\frac{\sqrt{x^2+a^2}}{a} + \frac{x}{a}\right] + C_1$$

$$= \ln\left[x + \sqrt{x^2+a^2}\,\right] + C \qquad (C = C_1 - \ln a)$$

例18. 求 $\int \frac{\mathrm{d}x}{\sqrt{x^2-a^2}} \quad (a>0).$

解: 当 $x>a$ 时, 令 $x = a\sec t,\ t\in(0,\frac{\pi}{2})$, 则

$$\sqrt{x^2-a^2} = \sqrt{a^2\sec^2 t - a^2} = a\tan t$$

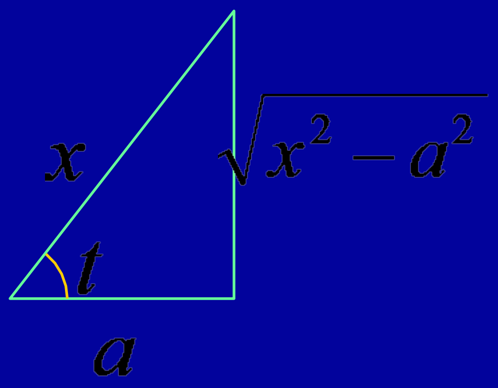
$$\mathrm{d}x = a\sec t\tan t\,\mathrm{d}t$$

$$\therefore\ 原式 = \int \frac{a\sec t\tan t}{a\tan t}\mathrm{d}t = \int \sec t\,\mathrm{d}t$$

$$= \ln\left|\sec t + \tan t\right| + C_1$$

$$= \ln\left|\frac{x}{a} + \frac{\sqrt{x^2-a^2}}{a}\right| + C_1$$

$$= \ln\left|x + \sqrt{x^2-a^2}\right| + C \quad (C = C_1 - \ln a)$$



当$x<-a$时，令 $x=-u$ 则$u>a$，于是

$$\int\frac{dx}{\sqrt{x^2-a^2}}=-\int\frac{du}{\sqrt{u^2-a^2}}=-\ln\left|u+\sqrt{u^2-a^2}\right|+C_1$$

$$=-\ln\left|-x+\sqrt{x^2-a^2}\right|+C_1$$

$$=-\ln\left|\frac{a^2}{-x-\sqrt{x^2-a^2}}\right|+C_1$$

$$=\ln\left|x+\sqrt{x^2-a^2}\right|+C \quad (C=C_1-2\ln a)$$

$x>a$时，$\int\frac{dx}{\sqrt{x^2-a^2}}=\ln\left|x+\sqrt{x^2-a^2}\right|+C$

例19. 求 $\int \frac{\sqrt{a^2-x^2}}{x^4}\mathrm{d}x$.

解: 令 $x=\frac{1}{t}$,则 $\mathrm{d}x=\frac{-1}{t^2}\mathrm{d}t$ （倒代换）

$$原式=\int \frac{\sqrt{a^2-\frac{1}{t^2}}}{\frac{1}{t^4}}\cdot\frac{-1}{t^2}\mathrm{d}t=-\int (a^2t^2-1)^{\frac{1}{2}}|t|\mathrm{d}t$$

当 $x>0$ 时,

$$原式=-\frac{1}{2a^2}\int (a^2t^2-1)^{\frac{1}{2}}\mathrm{d}(a^2t^2-1)$$

$$=-\frac{(a^2t^2-1)^{\frac{3}{2}}}{3a^2}+C=-\frac{(a^2-x^2)^{\frac{3}{2}}}{3a^2x^3}+C$$

当 $x<0$ 时, 类似可得一样结果 .

小结:

1. 第二类换元法常见类型:

(1) $\int f(x, \sqrt[n]{ax+b})\mathrm{d}x$, 令 $t=\sqrt[n]{ax+b}$

(2) $\int f(x, \sqrt[n]{\frac{ax+b}{cx+d}})\mathrm{d}x$, 令 $t=\sqrt[n]{\frac{ax+b}{cx+d}}$

（(1)(2) 第四节讲）

(3) $\int f(x, \sqrt{a^2-x^2})\mathrm{d}x$, 令 $x=a\sin t$ 或 $x=a\cos t$

(4) $\int f(x, \sqrt{a^2+x^2})\mathrm{d}x$, 令 $x=a\tan t$ 或 $x=a\cot t$

(5) $\int f(x, \sqrt{x^2-a^2})\mathrm{d}x$, 令 $x=a\sec t$ 或 $x=a\csc t$

(6) $\int f(a^x)\mathrm{d}x$, 令 $t = a^x$

(7) 分母中因子次数较高时, 可试用**倒代换**

2. 惯用基本积分公式补充 (P205)

(16) $\int \tan x \,\mathrm{d}x = -\ln|\cos x| + C$

(17) $\int \cot x \mathrm{d}x = \ln|\sin x| + C$

(18) $\int \sec x \mathrm{d}x = \ln|\sec x + \tan x| + C$

(19) $\int \csc x \mathrm{d}x = \ln|\csc x - \cot x| + C$

$$(20)\quad \int \frac{1}{a^2+x^2}\,\mathrm{d}x = \frac{1}{a}\arctan\frac{x}{a}+C$$

$$(21)\quad \int \frac{1}{x^2-a^2}\,\mathrm{d}x = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right|+C$$

$$(22)\quad \int \frac{1}{\sqrt{a^2-x^2}}\,\mathrm{d}x = \arcsin\frac{x}{a}+C$$

$$(23)\quad \int \frac{1}{\sqrt{x^2+a^2}}\,\mathrm{d}x = \ln(x+\sqrt{x^2+a^2})+C$$

$$(24)\quad \int \frac{1}{\sqrt{x^2-a^2}}\,\mathrm{d}x = \ln\left|x+\sqrt{x^2-a^2}\right|+C$$

例20. 求 $\int \frac{dx}{x^2+2x+3}$.

解: 原式 $= \int \frac{1}{(x+1)^2+(\sqrt{2})^2} d(x+1)$

$= \frac{1}{\sqrt{2}} \arctan \frac{x+1}{\sqrt{2}} + C$ (P205公式 (20))

例21. 求 $I = \int \frac{dx}{\sqrt{4x^2+9}}$.

解: $I = \frac{1}{2}\int \frac{d(2x)}{\sqrt{(2x)^2+3^2}} = \frac{1}{2}\ln(2x+\sqrt{4x^2+9})+C$

(P205公式 (23))

例22. 求 $\int \frac{\mathrm{d}x}{\sqrt{1+x-x^2}}$.

解：原式 $=\int \frac{\mathrm{d}(x-\frac{1}{2})}{\sqrt{(\frac{\sqrt{5}}{2})^2-(x-\frac{1}{2})^2}}=\arcsin\frac{2x-1}{\sqrt{5}}+C$

(P205 公式 (22))

例23. 求 $\int \frac{\mathrm{d}x}{\sqrt{e^{2x}-1}}$.

解：原式 $=-\int \frac{\mathrm{d}e^{-x}}{\sqrt{1-e^{-2x}}}=-\arcsin e^{-x}+C$

(P205 公式 (22))

例24. 求 $\int \frac{\mathrm{d}x}{x^2\sqrt{x^2+a^2}}$.

解: 令 $x=\frac{1}{t}$, 得

$$原式=-\int \frac{t}{\sqrt{a^2t^2+1}}\,\mathrm{d}t$$

$$=-\frac{1}{2a^2}\int \frac{\mathrm{d}\,(a^2t^2+1)}{\sqrt{a^2t^2+1}}=-\frac{1}{a^2}\sqrt{a^2t^2+1}+C$$

$$=-\frac{\sqrt{x^2+a^2}}{a^2x}+C$$

例25. 求 $\int \frac{\mathrm{d}x}{(x+1)^3\sqrt{x^2+2x}}$.

解: 原式 $= \int \frac{\mathrm{d}x}{(x+1)^3\sqrt{(x+1)^2-1}}$

令 $x+1=\frac{1}{t}$

$$= \int \frac{t^3}{\sqrt{\frac{1}{t^2}-1}}\left(-\frac{1}{t^2}\right)\mathrm{d}t = -\int \frac{t^2}{\sqrt{1-t^2}}\mathrm{d}t$$

$$= \int \frac{(1-t^2)-1}{\sqrt{1-t^2}}\mathrm{d}t = \int \sqrt{1-t^2}\,\mathrm{d}t - \int \frac{1}{\sqrt{1-t^2}}\mathrm{d}t$$

$$= \frac{1}{2}t\sqrt{1-t^2} + \frac{1}{2}\arcsin t - \arcsin t + C$$

$$= \frac{1}{2}\frac{\sqrt{x^2+2x}}{(x+1)^2} - \frac{1}{2}\arcsin\frac{1}{x+1} + C$$

思索与练习

1. 以下积分应怎样换元才使积分简便？

(1) $\int \frac{x^5}{\sqrt{1+x^2}}\mathrm{d}x$

令 $t=\sqrt{1+x^2}$

(2) $\int \frac{\mathrm{d}x}{\sqrt{1+e^x}}$

令 $t=\sqrt{1+e^x}$

(3) $\int \frac{\mathrm{d}x}{x(x^7+2)}$

令 $t=\frac{1}{x}$

2. 求以下积分:

1) $\int x^2 \frac{1}{\sqrt{x^3+1}}\mathrm{d}x = \frac{1}{3}\int \frac{1}{\sqrt{x^3+1}}\mathrm{d}(x^3+1)$

$$= \frac{2}{3}\sqrt{x^3+1} + C$$

2) $\int \frac{2x+3}{\sqrt{1+2x-x^2}}\mathrm{d}x = \int \frac{-(2-2x)+5}{\sqrt{1+2x-x^2}}\mathrm{d}x$

$$= -\int \frac{\mathrm{d}(1+2x-x^2)}{\sqrt{1+2x-x^2}} + 5\int \frac{\mathrm{d}(x-1)}{\sqrt{2-(x-1)^2}}$$

$$= -2\sqrt{1+2x-x^2} + 5\arcsin\frac{x-1}{\sqrt{2}} + C$$

3. 求不定积分 $\int \frac{2\sin x\cos x\sqrt{1+\sin^2 x}}{2+\sin^2 x}\mathrm{d}x$.

解：利用凑微分法，得

$$\text{原式} = \int \frac{\sqrt{1+\sin^2 x}}{2+\sin^2 x}\,\mathrm{d}(1+\sin^2 x)$$

令 $t=\sqrt{1+\sin^2 x}$

$$= \int \frac{2t^2}{1+t^2}\mathrm{d}t = 2\int (1-\frac{1}{1+t^2})\mathrm{d}t$$

$$= 2t - 2\arctan t + C$$

$$= 2[\sqrt{1+\sin^2 x} - \arctan\sqrt{1+\sin^2 x}\,] + C$$

4. 求不定积分 $\int \frac{1}{(1+x^2)\sqrt{1-x^2}}dx.$

解: 令 $x=\sin t$, $1+x^2=1+\sin^2 t$, $dx=\cos t\,dt$

原式 $=\int \frac{\cos t}{(1+\sin^2 t)\cos t}dt=\int \frac{1}{1+\sin^2 t}dt$

分子分母同除以 $\cos^2 t$

$=\int \frac{\sec^2 t}{\sec^2 t+\tan^2 t}dt=\int \frac{1}{1+2\tan^2 t}d\tan t$

$=\frac{1}{\sqrt{2}}\int \frac{1}{1+(\sqrt{2}\tan t)^2}d\sqrt{2}\tan t$

$=\frac{1}{\sqrt{2}}\arctan(\sqrt{2}\tan t)+C=\frac{1}{\sqrt{2}}\arctan\frac{\sqrt{2}x}{\sqrt{1-x^2}}+C$

例1. 求 $\int x\cos x\,\mathrm{d}x$.

$$\int uv'\,\mathrm{d}x = uv - \int u'v\,\mathrm{d}x$$

解: 令 $u = x$, $v' = \cos x$,

则 $u' = 1$, $v = \sin x$

$\therefore$ 原式 $= x\sin x - \int \sin x\,\mathrm{d}x$

$= x\sin x + \cos x + C$

思索: 怎样求 $\int x\sin x\,\mathrm{d}x$, $\int x^2\sin x\,\mathrm{d}x$?

提醒: 令 $u = x^2$, $v' = \sin x$, 则

原式 $= -x^2\cos x + 2\int x\cos x\,\mathrm{d}x$