Outline

1. Introduction of Frequency Response — Concept
2. Frequency Response of the Typical Elements
3. Bode Diagram of Open-loop System
4. Nyquist-criterion

} Graphical method

5. Frequency Response Based System Analysis
6. Frequency Response of Closed-loop Systems

} Analysis

6-1 Introduction

Key problem of control: stability and system performance

Question: Why Frequency Response

Analysis:

1. **Weakness of root locus method relies on the existence of open-loop transfer function**
2. **Weakness of time-domain analysis method is that time response is very difficult to obtain**
   - **Computational complex**
   - **Difficult for higher order system**
   - **Difficult to partition into main parts**
   - **Not easy to show the effects by graphical method**

Frequency Response Analysis

Three advantages:

* Frequency response(mathematical modeling) can be obtained directly by experimental approaches.
* Easy to analyze effects of the system with sinusoidal signals
* Convenient to measure system sensitivity to noise and parameter variations

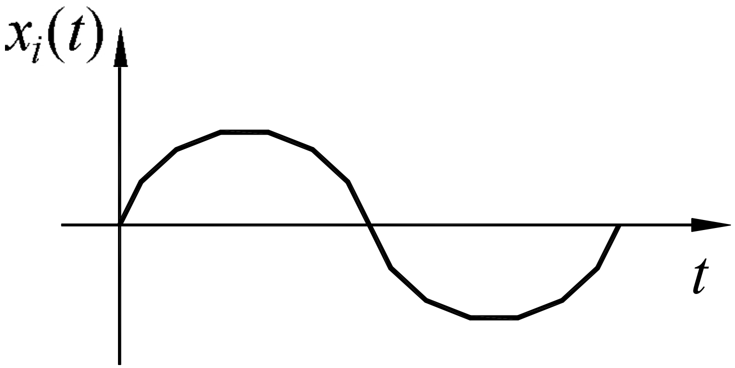
However, NEVER be limited to sinusoidal input

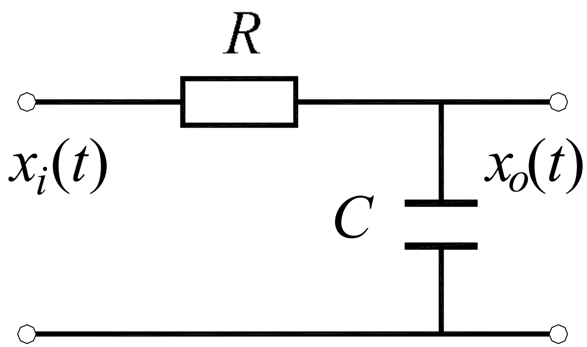
* Frequency-domain performances → time-domain performances

Frequency domain analysis is a kind of (indirect method) engineering method. It studies the system based on frequency response which is also a kind of mathematical model.

Frequency Response

Example 5.1: RC circuit





$$u_i = A\sin(\omega t + \omega_0)$$

$$U_i(s) = \frac{U_{1m}\omega}{s^2 + \omega^2} \qquad U_0(s) = \frac{1}{Ts+1} \cdot \frac{U_{1m}\omega}{s^2 + \omega^2}$$

Transfer function

$$G(s) = \frac{1}{RCs+1} = \frac{1}{Ts+1}$$

By using inverse Laplace transform

$$u_0(t) = \underbrace{\frac{U_{1m}T\omega}{1+T^2\omega^2} e^{\# \frac{t}{T}}}_{\text{Transient response}} + \underbrace{\frac{U_{1m}}{\sqrt{1+T^2\omega^2}} \sin(\omega t + !)}_{\text{Steady state response}} \qquad \varphi = \text{-arc}tg(! \, T)$$

Transient response **Steady state response**

Steady state response of u_o

$$\lim_{t \to \#} u_0 = \frac{U_{1m}}{\sqrt{1+T^2\omega^2}} \sin(\omega t + !)$$

$$= U_{1m} \left| \frac{1}{1+j!\,T} \right| \sin(!\,t + \angle \frac{1}{1+j!\,T})$$

Proposition: When the input to a linear time-invariant (LTI) system is sinusoidal, the steady-state output is a sinusoid with the same frequency but possibly with different amplitude and phase.

Definition：Frequency response (or characteristic) **is the ratio of the complex vector of the steady-state output versus sinusoidal input for a linear system.**

$$G(j!\,) = \frac{1}{1+j!\,T} = A(!\,)e^{j\varphi(!\,)}$$

$$A(!\,) = \left|\frac{1}{1+j!\,T}\right| = \frac{1}{\sqrt{1+!\,^2T^2}};$$

Magnitude response

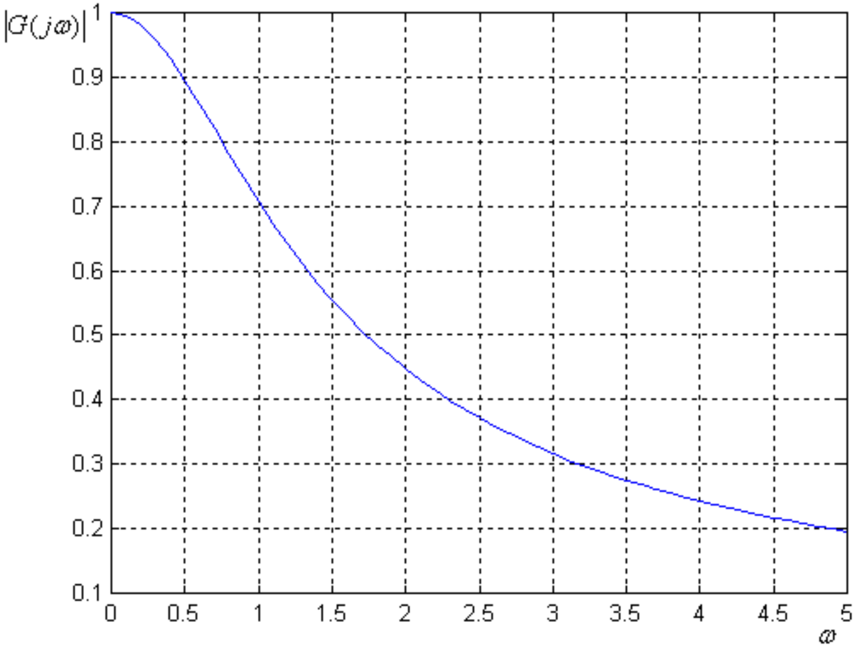
$$\phi(\omega) = \#\,(\frac{1}{1+j\omega T}) = -\mathrm{arctg}\,\omega T$$

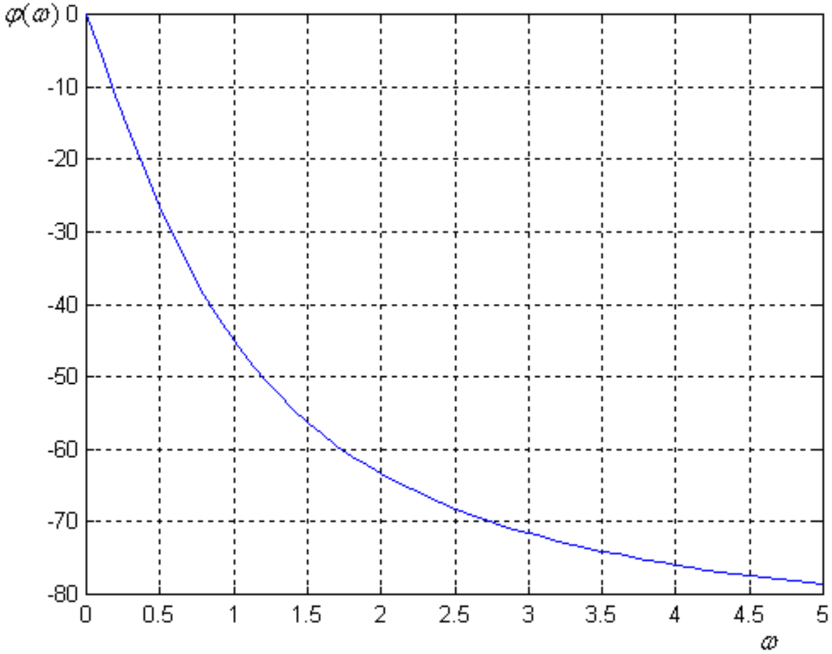
Phase response

$! \;= 0$ **The output has same magnitude and phase with input**

Magnitude will be attenuated and phase lag is increased.

$\omega\,!$





Magnitude of output versus input

Phase error of output and input

Magnitude characteristic

Phase characteristic

Generalized to linear time-invariant system

Transfer function of closed-loop system

$$G(s) = \frac{C(s)}{R(s)} = \frac{N(s)}{D(s)} = \frac{N(s)}{(s+p_1)(s+p_2)\cdots(s+p_n)}$$

where $p_1, \cdots, p_n$ are different closed-loop poles.

Given the sinusoidal input

$$r(t) = A_r \sin \omega t \qquad R(s) = \frac{A_r \omega}{s^2 + \omega^2}$$

$$C(s) = G(s) \cdot \frac{A_r \omega}{s^2 + \omega^2} = \frac{N(s)}{D(s)} \cdot \frac{A_r \omega}{s^2 + \omega^2}$$

$$= \frac{a}{s + j\omega} + \frac{\bar{a}}{s - j\omega} + \frac{b_1}{s + p_1} + \frac{b_2}{s + p_2} + \ldots + \frac{b_n}{s + p_n}$$

$$c(t) = ae^{-j\omega t} + \bar{a}e^{j\omega t} + b_1 e^{-p_1 t} + b_2 e^{-p_2 t} + \ldots + b_n e^{-p_n t}$$

$$= \sum_{i=1}^{n} b_i e^{-p_i t} + (ae^{-j\omega t} + \bar{a}e^{j\omega t})$$

$$= c_t(t) + c_s(t) \qquad (t \geq 0)$$

Transient response **Steady state response**

For a stable closed-loop system, we have $-p_i < 0$

$$a = G(s) \cdot \frac{A_r \omega}{s^2 + \omega^2} \cdot (s + j\omega)\big|_{s=-j\omega} = -\frac{A_r G(-j\omega)}{2j}$$

$$\bar{a} = G(s) \cdot \frac{A_r \omega}{s^2 + \omega^2} \cdot (s - j\omega)\big|_{s=j\omega} = \frac{A_r G(j\omega)}{2j}$$

$$G(j\omega) = |G(j\omega)| e^{j\angle G(j\omega)} \qquad G(-j\omega) = |G(-j\omega)| e^{-j\angle G(j\omega)} = |G(j\omega)| e^{-j\angle G(j\omega)}$$

Furthermore, we have

$$c_s(t) = a e^{-j\omega t} + \bar{a} e^{j\omega t}$$

$$= A_r |G(j\omega)| \frac{e^{j(\omega t + \angle G(j\omega))} - e^{-j(\omega t + \angle G(j\omega))}}{2j}$$

$$= A_r |G(j\omega)| \sin(\omega t + \angle G(j\omega))$$

$$= A_c \sin(\omega t + \varphi)$$

The magnitude and phase of steady state are as follows

$$A_c = A_r |G(j\omega)|; \qquad \varphi = \angle G(j\omega)$$

By knowing the transfer function G(s) of a linear system, the magnitude and phase characteristics completely describe the steady-state performance when the input is sinusoid.

Frequency-domain analysis can be used to predict both time-domain transient and steady-state system performance.

Relation of transfer function and frequency characteristic of LTI system (only for LTI system)

$$G(s)\big|_{s=j\omega} = G(j\omega) = |G(j\omega)|\, e^{j\angle G(j\omega)}$$

- Substitute $s=j\omega$ into the transfer function

$$F(s) = L[f(t)] = \int_{-\infty}^{+\infty} f(t)e^{-st}dt$$

Laplace transform

$$f(t) = L^{-1}[F(s)] = \int_{c-\infty}^{c+\infty} F(s)e^{st}ds$$

Inverse Laplace transform

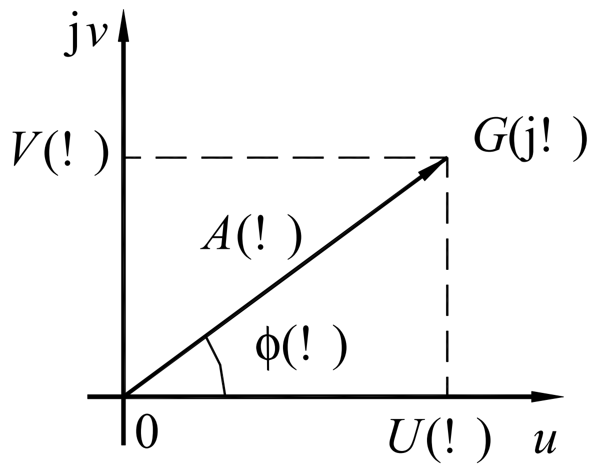
$$F(j\omega) = F\left[f(t)\right] = \int_{-\infty}^{+\infty} f(t)e^{-j\omega t}dt$$

Fourier transform

$$f(t) = F^{-1}\left[F(s)\right] = \frac{1}{2\pi j}\int_{-\infty}^{+\infty} F(j\omega)e^{j\omega t}d\omega$$

Inverse Fourier transform

Vector of frequency characteristics



It is a complex vector, and has three forms:

Algebraic form

$$G(j\omega)=U(\omega)+jV(\omega)$$

Polar form

$$G(j\omega)=\left|G(j\omega)\right|\angle G(j\omega)=A(\omega)\angle\phi(\omega)$$

Exponential form

$$G(j\omega)=\left|G(j\omega)\right|e^{j\angle G(j\omega)}=A(\omega)e^{j\phi(\omega)}$$

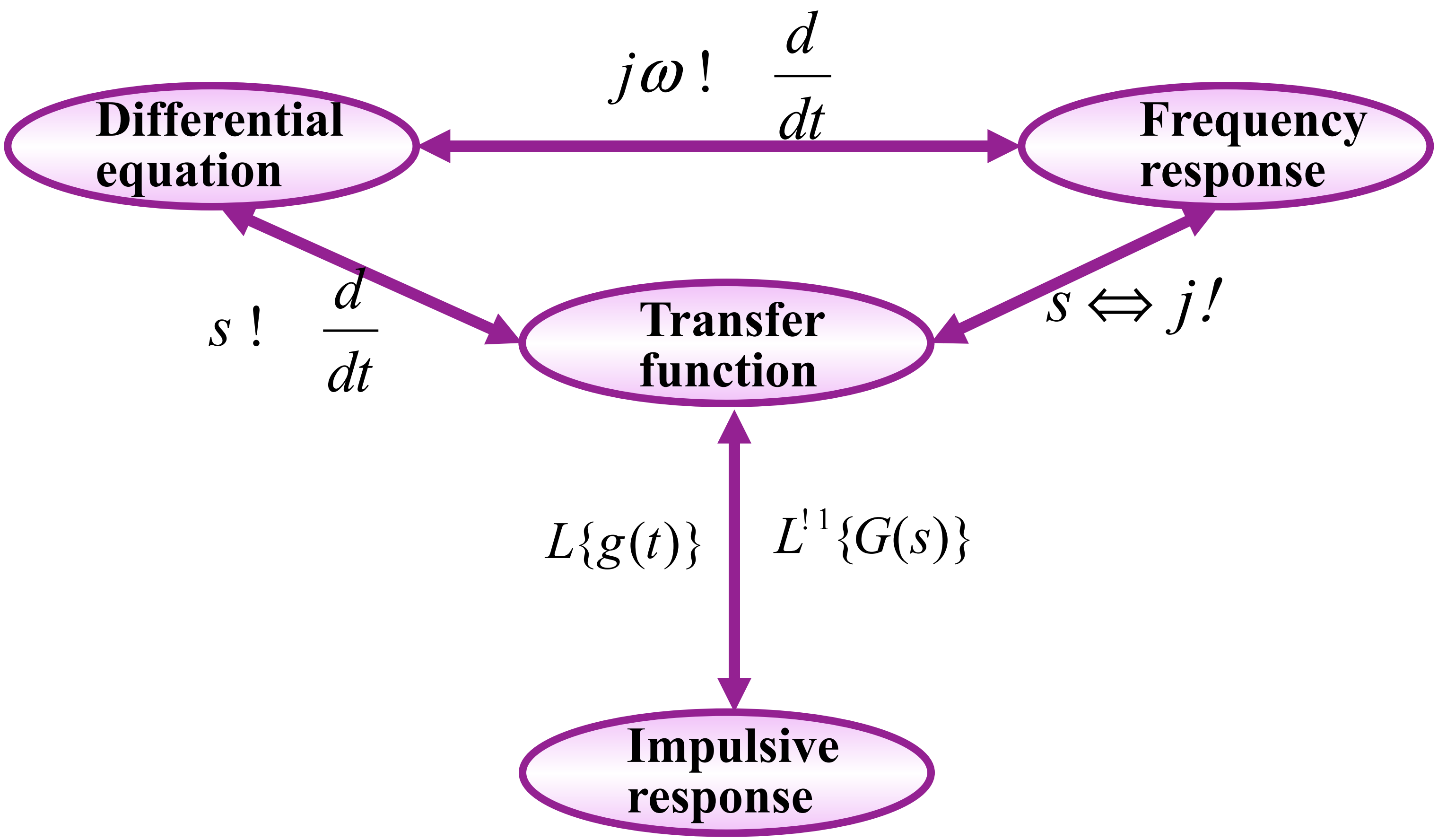
$$A(\omega)=\left|G(j\omega)\right|=\sqrt{U^2(\omega)+V^2(\omega)}$$

$$U(\omega)=A(\omega)\cos\phi(\omega)$$

$$\phi(\omega)=\arctan\left[\frac{V(\omega)}{U(\omega)}\right]$$

$$V(\omega)=A(\omega)\sin\phi(\omega)$$

We have learned following mathematical models：
differential equation, transfer function and frequency response



Example 5.2: Given the transfer function

$$G(s)=\frac{C(s)}{R(s)}=\frac{1}{s^2+3s+4}$$

Differential equation:

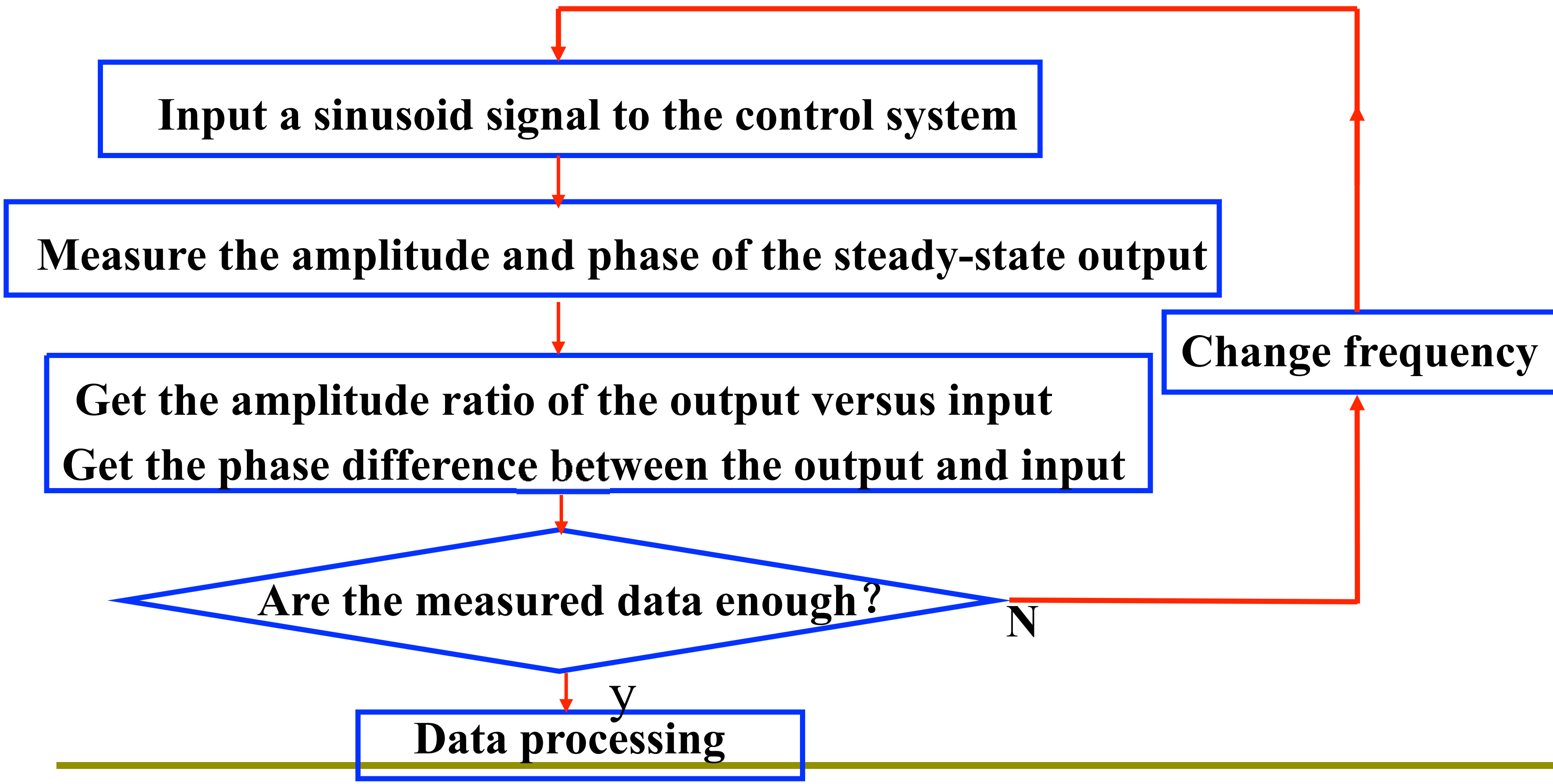
$$\frac{d^2c(t)}{dt^2}+3\frac{dc(t)}{dt}+4c(t)=r(t)$$

Frequency response:

$$G(j\omega)=\frac{c(j\omega)}{s(j\omega)}=\frac{1}{(j\omega)^2+3(j\omega)+4}=\frac{1}{4-\omega^2+3j\omega}$$

6-2 Frequency Characteristics of Typical Elements of system

How to get frequency characteristic?



Nyquist Diagram

- **Magnitude characteristic diagram**

  ***A*-*!* plot**
- **Phase characteristic diagram**

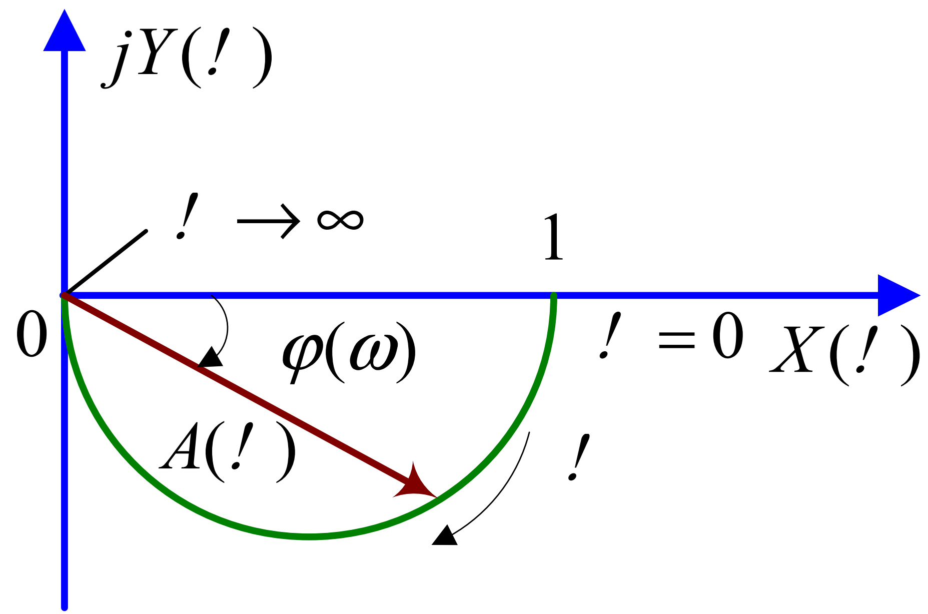
  [illegible]-*!* plot

  (Magnitude and phase diagrams: **Bode diagram**)
- **Gain-phase characteristic diagram** — **Nyquist diagram**

Polar form or algebraic form: *A* and [illegible] define a vector for a particular frequency *!* .

$$G(j\omega) = |G(j\omega)| e^{j\varphi(\omega)} = X(\omega) + jY(\omega)$$

!	0	1/(2″)	1/″	2/″	3/″	4/″	5/″	∞
A(!)	1	0.89	0.707	0.45	0.32	0.24	0.2	0
[illegible](!)	0	-26.6	-45	-63.5	-71.5	-76	-78.7	-90



Nyquist Diagram of RC Circuit

Bode Diagram

- **Bode Diagram: Logarithmic plots of magnitude response and phase response**
- **Horizontal axis: $\lg\omega$ (logarithmic scale to the base of 10) (unit: rad/s)**
- **Log Magnitude**

In feedback-system, the unit commonly used for the logarithm of the magnitude is the decibel (dB)

$$L(\omega) = 20\lg\left|G(j\omega)\right| = 20\lg A(\omega)$$

Property 1: As the frequency doubles, the decibel value increases by 6 dB.
As the frequency increases by a factor of 10, the decibel value increases by 20 dB.