# HG2ME2
# Mathematical Techniques for Electrical and Electronic Engineers 2

Section 4
Introduction to Partial Differential Equations (PDEs)



## Outline



1 Introduction

2 Laplace's and Poisson's Equation

3 The Heat Equation

4 The Wave Equation

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1. Introduction
2. Laplace's and Poisson's Equation
3. The Heat Equation
4. The Wave Equation




## Introduction

- Partial differential equations (PDEs) describe the behaviour of a dependent variable in terms of two or more independent variables.
- This is a generalization of an ODE in the case where we have more than one spatial variable and possibly a time variable.
- In this topic we introduce a classification of PDEs and discuss three of the simplest PDEs.
- There are numerous techniques for solving PDEs, but we will consider only a few here. Further modules will cover their solution in more detail.
- Relevant PDE examples include:
  - The wave equation.
  - Laplace's equation.
  - Maxwell's equations.
  - Navier-Stokes equations.
  - Schrödinger Equation.

## PDE Classification

- Consider PDEs of the form

$$a\frac{\partial^2 u}{\partial x^2} + b\frac{\partial^2 u}{\partial xy} + c\frac{\partial^2 u}{\partial y^2} + d\frac{\partial u}{\partial x} + e\frac{\partial u}{\partial y} + fu = 0,$$

  where $a$, $b$, $c$, $d$ and $e$ are constants (not all of $a$, $b$,and $c$ zero).
- This equation is a linear, second order PDE.
- We say that the PDE is
  - elliptic if $b^2 < 4ac$,
  - hyperbolic if $b^2 > 4ac$,
  - parabolic if $b^2 = 4ac$
- Compare this to the equations for ellipses, hyperbolae and parabolae.

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0.$$

- The different types of PDE have very different behaviours.
- We consider one PDE of each type.



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### Laplace's Equation

Laplace's equation is the PDE that arises when we set the Laplacian equal to zero, *i.e.*

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0,$$

where $u(x, y, z)$ is some function we are interested in finding.

- In two dimensions, Laplace's equation is

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

- Solutions of Laplace's equations are called harmonic functions.
- Laplace's equation is an elliptic PDE, in fact, it is the prototypical elliptic PDE.







- Suppose we have an electric field $\mathbf{E}$ in the presence of some charge $\rho(x, y, z)$.
- $\mathbf{E}$ is the gradient of some scalar electric potential $\phi$.

$$\mathbf{E} = -\nabla\phi.$$

- In this case, from Gauss' law, we have that

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}.$$

- Hence,

$$-\nabla \cdot (\nabla\phi) = -\nabla^2\phi = \frac{\rho}{\varepsilon_0}.$$

- This is an example of Poisson's equation, *i.e.* the Laplacian is equal to some non-zero RHS.

$$\nabla^2\phi = f.$$

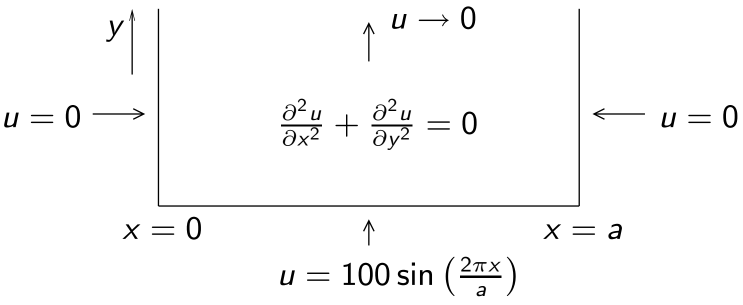
# Laplace's Equation - Boundary Conditions

- Suppose we wish to solve Laplace's equation on a domain $\Omega$ with boundary $\partial\Omega$.
- In order to be able to solve the equation we require boundary conditions to be imposed on $\partial\Omega$.
- Imposing the following types of boundary condition guarantee the existence of a unique solution of Laplace's equation:
  - Dirichlet boundary condition: $u$ is specified on all or part of the boundary.
  - Neumann boundary condition: The directional derivative perpendicular to the boundary $\frac{\partial u}{\partial \hat{\mathbf{n}}} = \nabla u \cdot \hat{\mathbf{n}}$ is specified on some part of the boundary. Here $\hat{\mathbf{n}}$ is the standard outward pointing unit normal.
  - Note, imposing a Neumann boundary condition on the whole of $\partial\Omega$ will not give a unique solution as $u + A$, for an arbitrary constant $A$, will also be a solution.
- The domain, $\partial\Omega$ need not be closed as we shall see in the next example.



# Laplace's Equation - Example

- Example.
  Suppose that $\Omega$ is the 2D domain bounded below by the line $y = 0$ and to the sides by the lines $x = 0$ and $x = a$. Furthermore, suppose that $u(x, y)$ is a function satisfying Laplace's equation and has Dirichlet boundary conditions as shown in the figure below.




We are given that the solution takes the form

$$u(x, y) = A \exp\left(-\frac{n\pi y}{a}\right) \sin\left(\frac{n\pi x}{a}\right),$$

with $n$ an integer.

1. Check that $u(x, y)$ satsifies Laplace's equations.
2. Deduce the constants $A$ and $n$ from the boundary conditions.

## Laplace's Equation - Example

1.

$$
\begin{aligned}
\frac{\partial u}{\partial x} &= A\left(\frac{n\pi}{a}\right)\exp\left(-\frac{n\pi y}{a}\right)\cos\left(\frac{n\pi x}{a}\right)\\
\frac{\partial^2 u}{\partial x^2} &= -A\left(\frac{n\pi}{a}\right)^2\exp\left(-\frac{n\pi y}{a}\right)\sin\left(\frac{n\pi x}{a}\right) = -\left(\frac{n\pi}{a}\right)^2 u\\
\frac{\partial u}{\partial y} &= -A\left(\frac{n\pi}{a}\right)\exp\left(-\frac{n\pi y}{a}\right)\sin\left(\frac{n\pi x}{a}\right)\\
\frac{\partial^2 u}{\partial y^2} &= A\left(\frac{n\pi}{a}\right)^2\exp\left(-\frac{n\pi y}{a}\right)\sin\left(\frac{n\pi x}{a}\right) = \left(\frac{n\pi}{a}\right)^2 u.
\end{aligned}
$$

Hence

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -\left(\frac{n\pi}{a}\right)^2 u + \left(\frac{n\pi}{a}\right)^2 u = 0.$$



## Laplace's Equation - Example

2. When $x = 0$

$$u(0, y) = A\exp\left(-\frac{n\pi y}{a}\right)\sin\left(\frac{n\pi 0}{a}\right) = 0$$

when $x = a$,

$$u(a, y) = A\exp\left(-\frac{n\pi y}{a}\right)\sin(n\pi) = 0.$$

So, the left and right boundary conditions are automatically satisfied.
When $y = 0$,

$$u(x, 0) = A\sin\left(\frac{n\pi x}{a}\right),$$

so to match the boundary conditions we require $A = 100$ and $n = 2$.
Finally, we show that $u$ satisifes the condition as $y \to \infty$.

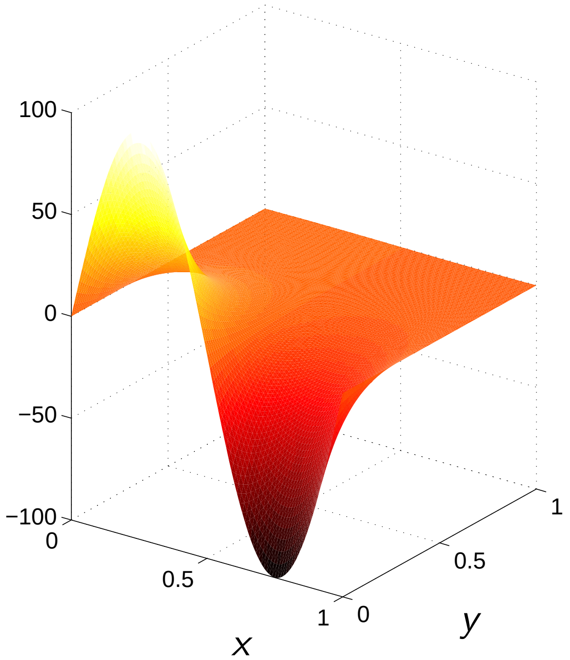
$$\lim_{y\to\infty} u(x, y) = 100\sin\left(\frac{2\pi x}{a}\right)\lim_{y\to\infty}\exp\left(-\frac{2\pi y}{a}\right) = 0$$

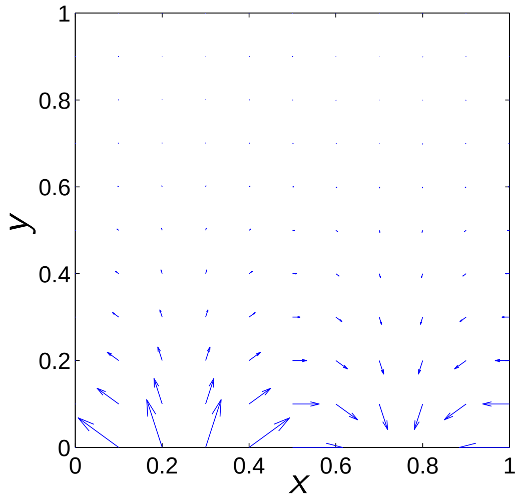
hence, all the boundary conditions are satisfied and

$$u(x, y) = 100\exp\left(-\frac{2\pi y}{a}\right)\sin\left(\frac{2\pi x}{a}\right).$$

# Laplace's Equation - Example

The two figures below show the potential $u$ and the resultant field $\mathbf{E} = \nabla u$, when $a = 1$.

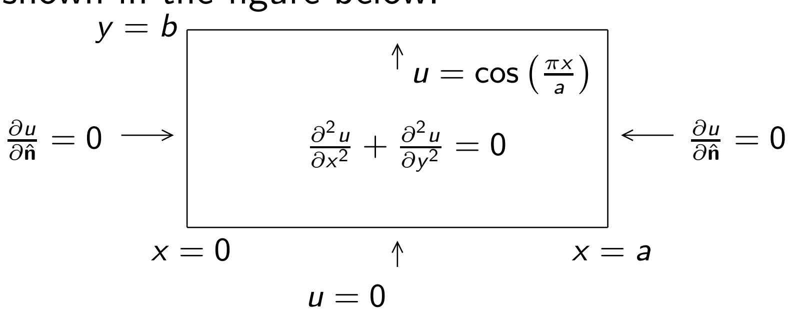





# Laplace's Equation - Example

- Example.
  Suppose now that $\Omega$ is the closed domain bounded by the lines $x = 0$, $x = a$, $y = 0$ and $y = b$ and furthermore that $u(x, y)$ satisfies Laplace's equation and has a combination of Dirichlet and Neumann conditions as shown in the figure below.




Assume that the solution has the form

$$u(x, y) = A + B \sinh\left(\frac{n\pi y}{a}\right) \cos\left(\frac{n\pi x}{a}\right),$$

with $n$ an integer.

1. Check that $u(x, y)$ satisfies Laplace's equation.
2. By using the boundary conditions, find $A$, $B$ and $n$.

## Laplace's Equation - Example

1.

$$\frac{\partial u}{\partial x} = -B\left(\frac{n\pi}{a}\right)\sinh\left(\frac{n\pi y}{a}\right)\sin\left(\frac{n\pi x}{a}\right)$$

$$\frac{\partial^2 u}{\partial x^2} = -B\left(\frac{n\pi}{a}\right)^2\sinh\left(\frac{n\pi y}{a}\right)\cos\left(\frac{n\pi x}{a}\right) = \left(\frac{n\pi}{a}\right)^2(A-u)$$

$$\frac{\partial u}{\partial y} = B\left(\frac{n\pi}{a}\right)\cosh\left(\frac{n\pi y}{a}\right)\cos\left(\frac{n\pi x}{a}\right)$$

$$\frac{\partial^2 u}{\partial y^2} = B\left(\frac{n\pi}{a}\right)^2\sinh\left(\frac{n\pi y}{a}\right)\cos\left(\frac{n\pi x}{a}\right) = \left(\frac{n\pi}{a}\right)^2(u-A)$$

Hence

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \left(\frac{n\pi}{a}\right)^2(A-u) + \left(\frac{n\pi}{a}\right)^2(u-A) = 0.$$



## Laplace's Equation - Example

2. On the left and right boundaries unit normals are $(-1,0)$ and $(1,0)$ respectively, so the Neumann boundary conditions are $\frac{\partial u}{\partial \mathbf{n}} = -\frac{\partial u}{\partial x} = 0$ and $\frac{\partial u}{\partial \mathbf{n}} = \frac{\partial u}{\partial x} = 0$, respectively.
When $x = 0$

$$\frac{\partial u}{\partial x}(0,y) = B\left(\frac{n\pi}{a}\right)\sinh\left(\frac{n\pi y}{a}\right)\sin\left(\frac{n\pi 0}{a}\right) = 0.$$

When $x = a$

$$\frac{\partial u}{\partial x}(a,y) = B\left(\frac{n\pi}{a}\right)\sinh\left(\frac{n\pi y}{a}\right)\sin(n\pi) = 0,$$

so the Neumann boundary conditions are automatically satisfied.
Now consider $y = 0$, we need $u(x,0) = 0$ and

$$u(x,0) = A + B\sinh\left(\frac{n\pi 0}{a}\right)\cos\left(\frac{n\pi x}{a}\right) = A = 0,$$

so $A = 0$.

## Laplace's Equation - Example

When $y = b$, $u(x,b) = \cos\left(\frac{n\pi}{a}\right)$, and

$$u(x,b) = B\sinh\left(\frac{n\pi b}{a}\right)\cos\left(\frac{n\pi x}{a}\right) = \cos\left(\frac{n\pi}{a}\right),$$

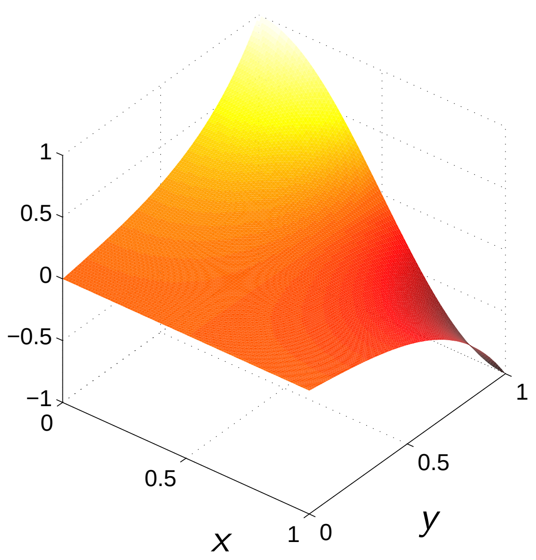
thus $n = 1$ and $B = \frac{1}{\sinh\left(\frac{\pi b}{a}\right)}$ and

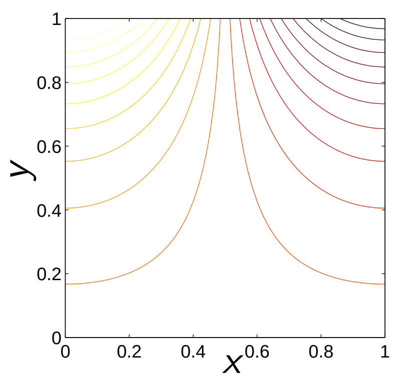
$$u(x,y) = \frac{\sinh\left(\frac{\pi y}{a}\right)\cos\left(\frac{\pi x}{a}\right)}{\sinh\left(\frac{\pi b}{a}\right)}.$$

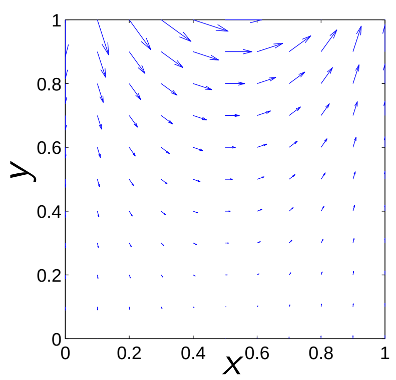
The figures on the next slide show the potential $u$ and the field $\mathbf{E} = \nabla u$ when $a = b = 1$.



## Laplace's Equation - Example

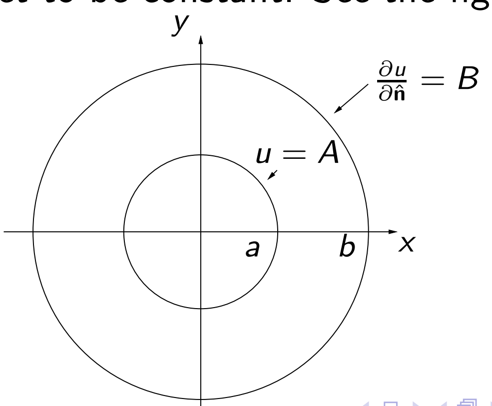


## Laplace's Equation and Symmetry

- We shall not go into much detail of how to actually solve Laplace's equation; this will be dealt with in future modules.
- We shall, however, show how symmetry can be used to reduce a PDE to an ODE.
- Suppose we are solving Laplace's equation on a domain which is the annulus centered on the origin and bounded by the circles $x^2 + y^2 = a^2$ and $x^2 + y^2 = b^2$ with $0 < a < b$.
- Suppose also that on the inner circle, $u$ is set to be a constant and on the outer circle $\frac{\partial u}{\partial \hat{\mathbf{n}}}$ is set to be constant. See the figure below.






## Laplace's Equation and Symmetry

- As the domain has axisymmetry we can rewrite the Laplacian in polar coordinates $(\rho, \varphi)$ and find $u(\rho, \varphi)$ such that

$$\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial u}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 u}{\partial \varphi^2} = 0.$$

- As the boundary conditions have no dependence on $\varphi$ we can assume that $u$ has no dependence on $\varphi$ either, so $u(\rho, \varphi) = u(\rho)$.
- This means $\frac{\partial u}{\partial \varphi} = \frac{\partial^2 u}{\partial \varphi^2} = 0$, hence the PDE becomes an ODE:

$$\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial u}{\partial \rho}\right) = 0.$$

with boundary conditions

$$u(a) = A, \quad \frac{\partial u}{\partial \rho}(b) = B.$$

- Example. Let $\Omega$ be the annulus enclosed by the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 2$. Suppose $u$ satisfies Laplace's equation and on the inner circle $u = 1$ and on the outer circle $\frac{\partial u}{\partial \hat{\mathbf{n}}} = 1$.
Find $u$.
We have that $\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial u}{\partial \rho}\right) = 0$, hence

$$\frac{\partial}{\partial \rho}\left(\rho \frac{\partial u}{\partial \rho}\right) = 0$$

Integrating gives

$$\rho \frac{\partial u}{\partial \rho} = C, \quad C \text{ a constant.}$$

Hence

$$\frac{\partial u}{\partial \rho} = \frac{C}{\rho}$$

Integrating again gives

$$u = C\ln(\rho) + D. \quad D \text{ a constant.}$$

We must find $C$ and $D$.





When $\rho = 1$, $\quad u(1) = C\ln(1) + D = D = 1,$

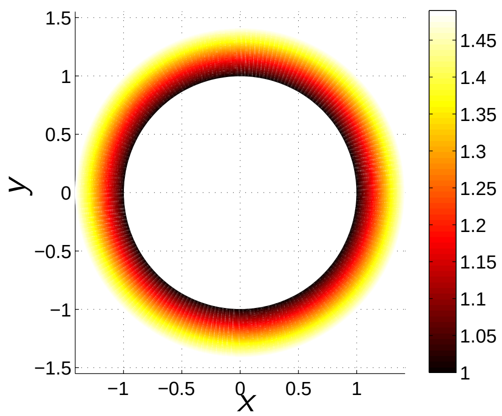
When $\rho = \sqrt{2}$, $\quad \frac{\partial u}{\partial \rho}(\sqrt{2}) = \frac{C}{\sqrt{2}} = 1,$

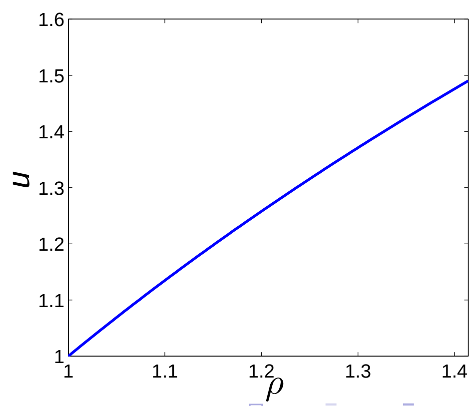
So,

$$u(\rho) = \sqrt{2}\ln(\rho) + 1.$$

Converting back to Cartesian coordinates we have

$$u(x, y) = \sqrt{2}\ln(\sqrt{x^2 + y^2}) + 1.$$



# Outline


1. Introduction
2. Laplace's and Poisson's Equation
3. The Heat Equation
4. The Wave Equation




# The Heat Equation

The University of Nottingham

- Suppose a block of metal is heated and then allowed to cool.
- Under the assumption that heat is transferred only by conduction, the temperature distribution $u(x, y, z, t)$ in the metal is given by the heat equation.

**Heat Equation**

$$\frac{\partial u}{\partial t} = \nabla^2 u.$$

Note, here $\nabla^2$ operates only with respect to the spatial variables $x$, $y$ and $z$

## The Heat Equation

- In one space dimension, the heat equation becomes

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}.$$

- Hence, the heat equation is a parabolic PDE.
- Suppose we wish to solve the equation on an interval $a < x < b$, for a time period $0 \leq t << T$.
- We must supplement the PDE with boundary conditions and an initial condition to find the solution

$$\left.\begin{aligned} u(a,t) &= f(t), \quad t > 0, \\ u(b,t) &= g(t), \quad t > 0, \end{aligned}\right\} \text{ Boundary Conditions.}$$

$$u(x,0) = w(x), \quad a \leq x \leq b, \qquad \text{Initial condition.}$$

- In a similar way to Laplace's equation, given a general form for a solution we must check it satisfies the equation by differentiating and also that it satisfies the boundary and initial conditions.
- Other approaches to solving the heat equation will not be considered here.



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