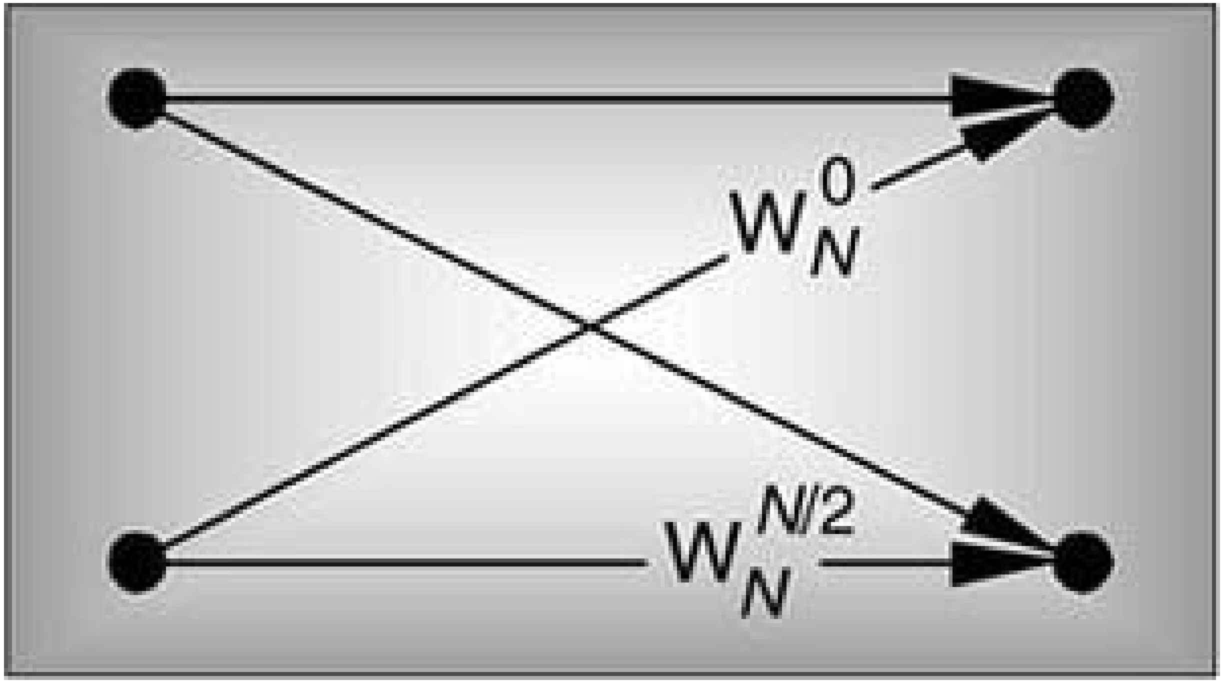
Chapter Four. The Fast Fourier Transform



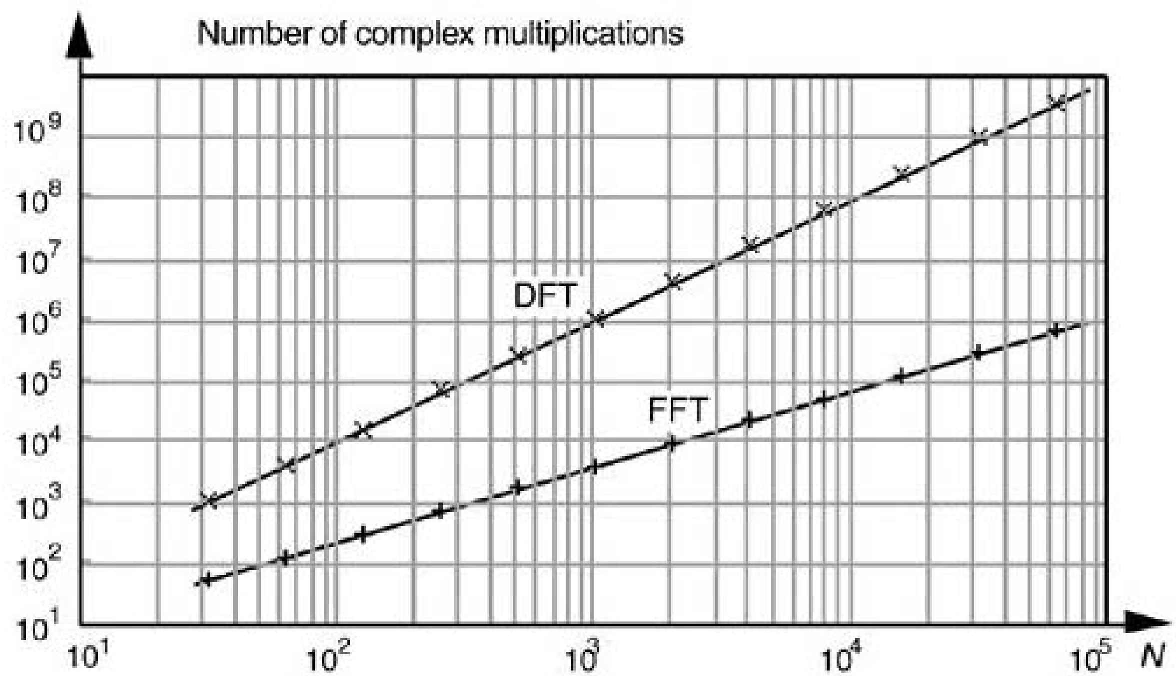
4.1. RELATIONSHIP OF THE FFT TO THE DFT

The expression for an N-point DFT,

$$X(m) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi nm/N}$$

For an N-point DFT, we'd have to perform N^2 complex multiplications.

Figure 4-1. Number of complex multiplications in the DFT and the radix-2 FFT as a function of N.



4.2. HINTS ON USING FFTS IN PRACTICE

4.2.1 Sample Fast Enough and Long Enough

data collection time interval must be long enough to satisfy our desired FFT frequency resolution for the given sample rate fs.

$$N = \frac{f_s}{desired\ \ resolution} \qquad f_{analysis}(m) = \frac{mf_s}{N}$$

4.2.2 Manipulating the Time Data Prior to Transformation

If sequence length is not an integral power of two, we have two options.

(1) To discard enough data samples so that the remaining FFT input sequence length is some integral power of two.

(2) To append enough zero-valued samples to the end of the time data sequence to match the number of points of the next largest radix-2 FFT

4.2.4 Interpreting FFT Results

Compute the absolute frequency of the individual FFT bin Centers.

the FFT bin spacing is the ratio of the sampling rate (fs) over the number of points in the FFT, or fs/N.

We can determine the true amplitude of time-domain signals from their FFT spectral results.

$$X(m) = X_{real}(m) + jX_{image}(m)$$

the FFT output magnitude samples,

$$X_{mag}(m) = |X(m)| = \sqrt{X_{real}(m)^2 + X_{image}(m)^2}$$

power spectrum $X_{PS}(m)$ of an FFT result

$$X_{PS}(m) = |X(m)|^2 = X_{real}(m)^2 + X_{image}(m)^2$$

The power spectrum in decibels

$$X_{db}(m) = 10\log_{10}\left(\left|X(m)\right|^2\right) dB$$

The normalized power spectrum in decibels can be calculated using

$$\text{normalized } X_{dB}(m) = 10\log_{10}\left(\frac{\left|X(m)\right|^2}{\left(\left|X(m)\right|_{\max}\right)^2}\right)$$

or

$$\text{normalized } X_{dB}(m) = 20\log_{10}\left(\frac{\left|X(m)\right|}{\left|X(m)\right|_{\max}}\right)$$

Normalization through division by $(|X(m)|_{\max})^2$ or $|X(m)|_{\max}$ eliminates the effect of any absolute FFT or window scale factors.

The phase angles $X_{ø}(m)$ of the individual FFT outputs are given by

$$X_{\phi}(m) = \tan^{-1}\left(\frac{X_{image}(m)}{X_{real}(m)}\right)$$

4.4. DERIVATION OF THE RADIX-2 FFT ALGORITHM

$x(n)$ is segmented into its even and odd indexed elements

$$X(m) = \sum_{n=0}^{(N/2)-1} x(2n) e^{-j2\pi(2n)m/N} + \sum_{n=0}^{(N/2)-1} x(2n+1) e^{-j2\pi(2n+1)m/N}$$

Pulling the constant phase angle outside the second summation,

$$X(m) = \sum_{n=0}^{(N/2)-1} x(2n) e^{-j2\pi(2n)m/N} + e^{-j2\pi m/N} \sum_{n=0}^{(N/2)-1} x(2n+1) e^{-j2\pi(2n)m/N}$$

We'll define $W_N = e^{-j2\pi/N}$ (twiddle factors) to represent the complex phase angle factor that is constant with N

$$X(m) = \sum_{n=0}^{N-1} x(n) W_N^{nm} = \sum_{n=0}^{(N/2)-1} x(2n) W_N^{2nm} + W_N^{m} \sum_{n=0}^{(N/2)-1} x(2n+1) W_N^{2nm}$$

the expression for an N-point DFT,

$$X(m) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi nm/N}$$

For an N-point DFT, we'd have to perform N^2 complex multiplications.

$$X(0) \rightarrow x(0)e^{-j2\pi.0.0/N}, x(1)e^{-j2\pi.1.0/N}, \mathrm{L}\ , x(N-1)e^{-j2\pi.(N-1).0/N}$$

$$X(1) \rightarrow x(0)e^{-j2\pi.0.1/N}, x(1)e^{-j2\pi.1.1/N}, \mathrm{L}\ , x(N-1)e^{-j2\pi.(N-1).1/N}$$

$$\mathrm{L}$$

$$X(m) \rightarrow x(0)e^{-j2\pi.0.m/N}, x(1)e^{-j2\pi.1.m/N}, \mathrm{L}\ , x(N-1)e^{-j2\pi.(N-1).m/N}$$

$$X(N-1) \rightarrow x(0)e^{-j2\pi.N-1/N}, x(1)e^{-j2\pi.2.N-1/N}, \mathrm{L}\ , x(N-1)e^{-j2\pi.(N-1)/N}$$

Because $W_N^2 = e^{-j2\pi 2/N} = e^{-j2\pi/(N/2)} = W_{N/2}$

$$X(m) = \sum_{n=0}^{(N/2)-1} x(2n) W_{N/2}^{nm} + W_N^m \sum_{n=0}^{(N/2)-1} x(2n+1) W_{N/2}^{nm}$$

Consider the $X(m+N/2)$ output. If we plug $m+N/2$ in for m

$$X(m+N/2) = \sum_{n=0}^{(N/2)-1} x(2n) W_{N/2}^{n(m+N/2)} + W_N^{(m+N/2)} \sum_{n=0}^{(N/2)-1} x(2n+1) W_{N/2}^{n(m+N/2)}$$

We can now simplify the phase angle terms

$$W_{N/2}^{n(m+N/2)} = W_{N/2}^{nm} W_{N/2}^{nN/2} = W_{N/2}^{nm} \left(e^{-\frac{j2\pi}{N/2}} \right)^{nN/2} = W_{N/2}^{nm} \left(e^{-j2n\pi} \right) = W_{N/2}^{nm}(1) = W_{N/2}^{nm}$$

For any integer n., we can simplify it as $\quad W_{N/2}^{N/2} = 1$

$$W_N^{(m+N/2)} = W_N^m W_N^{N/2} = W_N^m \left(e^{-j2\pi/N} \right)^{N/2} = W_{N/2}^m e^{-j\pi} = W_{N/2}^m(-1) = -W_N^m$$

We represent $X(m+N/2)$ as

$$X(m+N/2) = \sum_{n=0}^{(N/2)-1} x(2n) W_{N/2}^{nm} - W_N^m \sum_{n=0}^{(N/2)-1} x(2n+1) W_{N/2}^{nm}$$

See the similarity

$$X(m)=\sum_{n=0}^{(N/2)-1}x(2n)W_{N/2}^{nm}+W_N^m\sum_{n=0}^{(N/2)-1}x(2n+1)W_{N/2}^{nm}$$

$$X(m+N/2)=\sum_{n=0}^{(N/2)-1}x(2n)W_{N/2}^{nm}-W_N^m\sum_{n=0}^{(N/2)-1}x(2n+1)W_{N/2}^{nm}$$

For an N-point DFT, we perform an N/2-point DFT to get the first N/2 outputs and use those to get the last N/2 outputs.

Number of complex multiplications in the DFT : $N^2 \rightarrow 2(N/2)^2 = N^2/2$

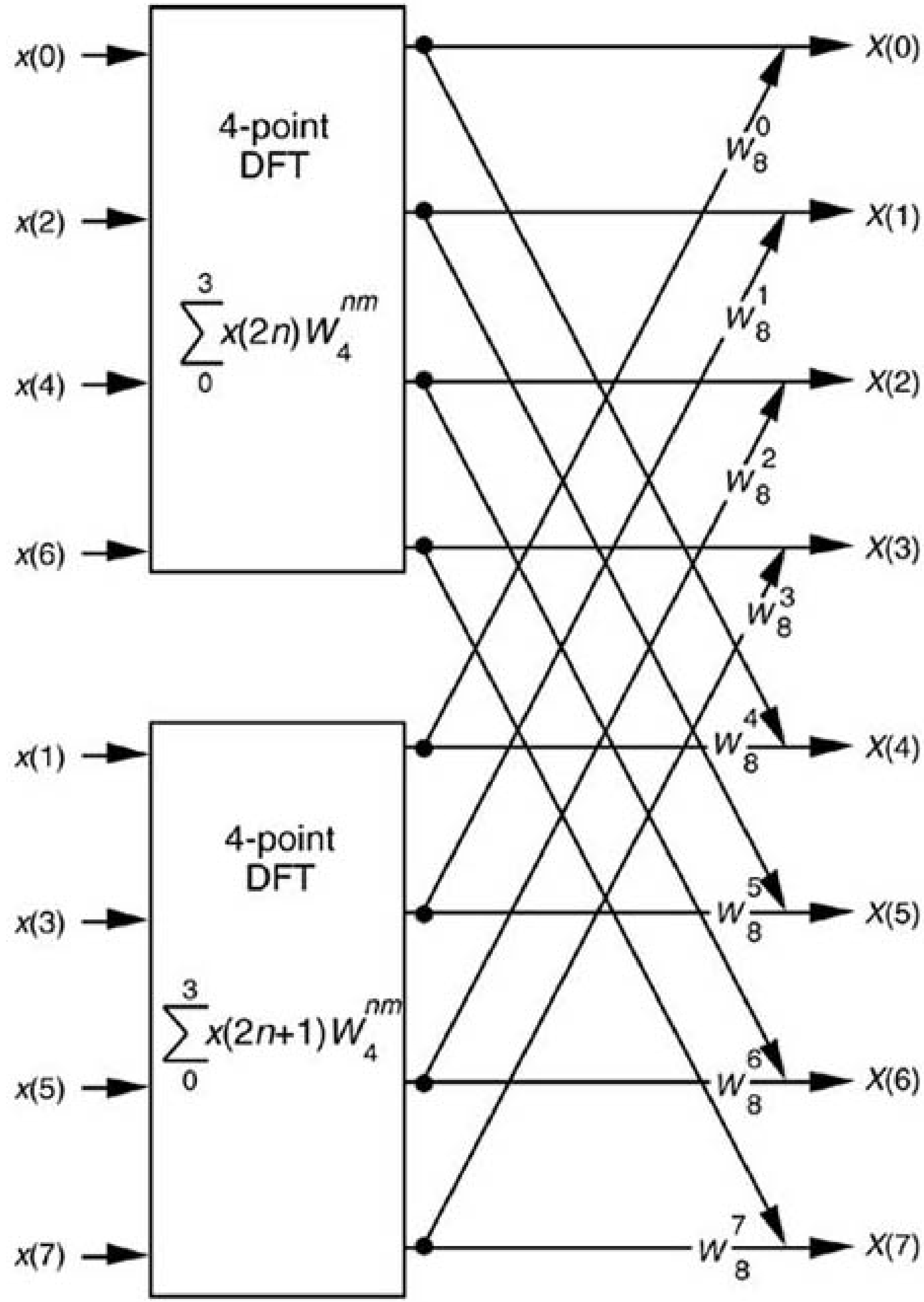
If we simplify the two equations above to the form

$$X(m)=A(m)+W_N^m B(m) \qquad A(m), B(m) \text{ are DFTs too}$$

$$X(m+N/2)=A(m)-W_N^m B(m)$$

We can think about breaking the two 4-point DFTs into four 2-point DFTs.
We segment the inputs to the upper 4-point DFT into their odd and even components

Figure 4-2. FFT implementation of an 8-point DFT using two 4-point DFTs.



$$X(m) = A(m) + W_N^m B(m)$$

$$X(m + N/2) = A(m) - W_N^m B(m)$$

$$X(0) = A(0) + W_8^0 B(0)$$

$$A(m) = \sum_{n=0}^{(N/2)-1} x(2n) W_{N/2}^{nm} = \sum_{n=0}^{(N/4)-1} x(4n) W_{N/2}^{2nm} + \sum_{n=0}^{(N/4)-1} x(4n+2) W_{N/2}^{(2n+1)m}$$

$$= \sum_{n=0}^{(N/4)-1} x(4n) W_{N/4}^{2nm} + W_{N/2}^{m} \sum_{n=0}^{(N/4)-1} x(4n+2) W_{N/4}^{2nm} \qquad W_{N/2}^{2nm} = W_{N/4}^{nm}$$

similarly $$B(m) = \sum_{n=0}^{(N/4)-1} x(4n) W_{N/4}^{2nm} - W_{N/2}^{m} \sum_{n=0}^{(N/4)-1} x(4n+1) W_{N/4}^{2nm}$$

$p = 2n$ $$A(m) = \sum_{p=0}^{(N/4)-1} x(2p) W_{N/4}^{pm} + W_{N/2}^{m} \sum_{p=0}^{(N/4)-1} x(2p+1) W_{N/4}^{pm}$$

$$B(m) = \sum_{p=0}^{(N/4)-1} x(2p) W_{N/4}^{pm} - W_{N/2}^{m} \sum_{p=0}^{(N/4)-1} x(2p+1) W_{N/4}^{pm}$$

For any N-point DFT, we can break each of the N/2-point DFTs into two N/4-point DFTs to further reduce the number of sine and cosine multiplications. Eventually, we would arrive at an array of 2-point DFTs where no further computational savings could be realized.

Figure 4-3. FFT implementation of an 8-point DFT as two 4-point DFTs and four 2-point DFTs.