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CHAPTER 6

Analysis of Completely Randomized Designs

In this chapter techniques for the analysis of data obtained from balanced, completely randomized designs are presented. Analysis-of-variance procedures are introduced as a methodology for the simultaneous assessment of factor effects. The analysis of complete factorial experiments in which factor effects are fixed (constant) is stressed. Multiple comparison procedures that are used to understand the magnitude and direction of individual and joint factor effects are also discussed. Adaptations of these procedures to unbalanced designs, to designs with nested factors, to blocking designs, and to alternative models are detailed in subsequent chapters. The major topics included in this chapter are:

- *analysis-of-variance decomposition of the variation of the observed responses into components due to assignable causes and to uncontrolled experimental error,*
- *estimation of model parameters for both qualitative and quantitative factors,*
- *statistical tests of factor effects, and*
- *multiple comparison techniques for comparing means or groups of means.*

Chapter 3 of this book details inference procedures for means and standard deviations from one or more samples. The analysis of data from many diverse types of experiments can be placed in this context. Many others, notably the multifactor experiments described in Chapters 4 and 5, require alternative inferential procedures. In this chapter we introduce general procedures for analyzing multifactor experiments.

To more easily focus on the concepts involved in the analysis of multifactor experiments, we restrict attention in this chapter to balanced complete factorial experiments involving factors whose effects on a response are, apart from uncontrollable experimental error, constant. Analytic techniques for data obtained from completely randomized designs are stressed. Thus, the topics covered in this chapter are especially germane to the types of experiments discussed in Chapter 5.

Analysis-of-variance procedures are introduced in Section 6.1 for multi- factor experiments. Estimation of analysis-of-variance model parameters in Section 6.2 includes interval estimation and the estimation of polynomial effects for quantitative factors. Statistical tests for factor effects are presented in Section 6.3. Multiple comparisons involving linear combinations of means is discussed in Section 6.4.

6.1 BALANCED MULTIFACTOR EXPERIMENTS

Single-factor experiments consist of a single controllable design factor whose levels are believed to affect the response. If the factor levels affect the mean of the response, the influences of the factor levels are referred to as *fixed* (constant) *effects*. If the factor levels affect the variability of the response, the factor effects are referred to as *random effects*.

Multifactor experiments consist of two or more controllable design factors whose *combinations* of levels are believed to affect the response. The number of levels can be different for each factor. Balanced complete factorial experi- ments have an equal number of repeat tests for all *combinations* of levels of the design factors. The torque study and the chemical pilot plant experiment introduced in Chapter 5 are examples of balanced multifactor experiments in which the factor effects are all fixed. In the following subsections, we intro- duce the analysis-of-variance procedures for multifactor experiments involving fixed effects. A single-factor experiment is an important special case that is illustrated with an example.

6.1.1 Fixed Factor Effects

Factor effects are called *fixed* if the levels of the factor are specifically selected because they are the only ones for which inferences are desired (see Exhibit 6.1). Because the factor levels can be specifically chosen, it is assumed that the factor levels exert constant (apart from experimental error) influences on the response. These effects are modeled by unknown parameters in sta- tistical models that relate the mean of the response variable to the factor levels (see Section 6.1.2). Fixed factor levels are not randomly selected nor are they the result of some chance mechanisms; they are intentionally chosen for investigation. This is the primary distinction between fixed and random factor levels (see Section 10.1).

EXHIBIT 6.1

Fixed Factor Effects. Factors have fixed effects if the levels chosen for inclusion in the experiment are the only ones for which inferences are desired.

The pilot-plant chemical-yield study introduced in Section 5.2 is an experiment in which all the factors have fixed effects. Table 6.1 lists the factor–level combinations and responses for this experiment. The two temperatures, the two concentrations, and the two catalysts were specifically chosen and are the only ones for which inferences can be made. One cannot infer effects on the chemical yield for factor levels that are not included in the experiment — for example, that the effect of a temperature of 170°C would be between those of 160 and 180°C — unless one is willing to make additional assumptions about the change in the mean response between the levels included in the experiment.

At times one is willing to assume, apart from experimental error, that there is a smooth functional relationship between the quantitative levels of a factor and the response. For example, one might be willing to assume that temperature has a linear effect on the chemical yield. If so, the fitting of a model using two levels of temperature would allow one to fit a straight line between the average responses for the two levels and infer the effect of a temperature of, say, 170°C. In this setting the levels of temperature are still fixed and inferences can only be drawn on the temperature effects at these two

TABLE 6.1 Test Results from Pilot-Plant Chemical-Yield Experiment*

Temperature (°C)	Concentration (%)	Catalyst	Yield (g)
160	20	C_1	59, 61
		C_2	50, 54
	40	C_1	50, 58
		C_2	46, 44
180	20	C_1	74, 70
		C_2	81, 85
	40	C_1	69, 67
		C_2	79, 81

*Two repeat test runs for each factor–level combination.

levels. It is the added assumption of a linear effect of temperature between 160 and 180°C that allows inferences to be made for temperatures between the two included in the experiment.

6.1.2 Analysis-of-Variance Models

Analysis-of-variance (ANOVA) procedures separate or partition the variation observable in a response variable into two basic components: variation due to assignable causes and to uncontrolled or random variation (see Exhibit 6.2). Assignable causes refer to known or suspected sources of variation from vari- ates that are controlled (experimental factors) or measured (covariates) during the conduct of an experiment. Random variation includes the effects of all other sources not controlled or measured during the experiment. While ran- dom sources of response variability are usually described as including only chance variation or measurement error, any factor effects that have been nei- ther controlled nor measured are also included in the measurement of the component labeled "random" variation.

EXHIBIT 6.2 SOURCES OF VARIATION

Assignable Causes. Due to changes in experimental factors or measured covariates.
Random Variation. Due to uncontrolled effects, including chance causes and measurement errors.

Assignable causes corresponding to controllable factors can be either fixed or random effects. Fixed effects were described in the last section. Random effects will be discussed in Part III of this text. A more complete and explicit description of factor effects requires the specification of a statistical model that includes both assignable causes and random variation. Analysis-of-variance models for experiments in which all factor effects are fixed adhere to the assumptions listed in Exhibit 6.3.

EXHIBIT 6.3 FIXED-EFFECTS MODEL ASSUMPTIONS

1. The levels of all factors in the experiment represent the only levels of which inferences are desired.
2. The analysis-of-variance model contains parameters (unknown constants) for all main effects and interactions of interest in the experiment.
3. The experimental errors are statistically independent.
4. The experimental errors are satisfactorily modeled by the normal probability distribution with mean zero and (unknown) constant standard deviation.

A statistical model for the pilot-plant chemical yield example is

$$y_{ijkl} = \mu_{ijk} + e_{ijkl}, \quad i = 1, 2, \quad j = 1, 2, \quad k = 1, 2, \quad l = 1, 2. \tag{6.1}$$

In this model, μ_{ijk} represents the effects of the assignable causes and e_{ijkl} represents the random error effects. The expression of the model as in Equation (6.1) stresses the connection between multifactor experiments with fixed effects and sampling from a number of populations or processes that differ only in location; that is, in their means. The assignable cause portion of the model can be further decomposed into terms representing the main effects and the interactions among the three model factors:

$$\mu_{ijk} = \mu + \alpha_i + \beta_j + \gamma_k + (\alpha\beta)_{ij} + (\alpha\gamma)_{ik} + (\beta\gamma)_{jk} + (\alpha\beta\gamma)_{ijk}. \tag{6.2}$$

The subscript i in the model (6.1) refers to one of the levels of temperature, the subscript j to one of the levels of concentration, the subscript k to one of the catalysts, and the subscript l to one of the two repeat tests. The components of (6.2) permit the mean of the response to be affected by both the individual factor levels (main effects) and combinations of two or more factor levels (interactions).

The parameters in (6.2) having a single subscript represent main effects for the factor identified by the subscript. Two-factor interactions are modeled by the terms having two subscripts, and the three-factor interaction by terms having three subscripts. The first term in equation (6.2) represents the overall response mean, that is, a term representing an overall population or process mean from which the various individual and joint factor effects are measured.

The interaction components of (6.2) model joint effects that cannot be suitably accounted for by the main effects. Each interaction effect measures the incremental joint contribution of factors to the response variability over the contribution of the main effects and lower-order interactions. Thus, a two-factor interaction is present in a model only if the effects of two factors on the response cannot be adequately modeled by the main effects. Similarly, a three-factor interaction is included only if the three main effects and the three two-factor interactions do not satisfactorily model the effects of the factors on the response.

The parameters representing the assignable causes in (6.2) can be expressed in terms of the model means μ_{ijk}. Specification of the right side of (6.2) must be carefully considered both to allow meaningful interpretation and because the right side is not unique. To illustrate the lack of uniqueness, for any two values of μ and α_i, both $\mu + \alpha_i$ and $\mu^* + \alpha_i^*$ with $\mu^* = \mu + c$ and $\alpha_i^* = \alpha_i - c$ give the same numerical value for μ_{ijk} regardless of the value of the constant c. For this representation to be unique, we must impose constraints on the values of the parameters. The commonly used constraints we impose are:

$$\sum_i \alpha_i = 0, \quad \sum_j \beta_j = 0, \quad \sum_k \gamma_k = 0, \quad \sum_i (\alpha\beta)_{ij} = 0, \quad \sum_j (\alpha\beta)_{ij} = 0, \text{ etc.}$$

Thus, the constraints require that each of the parameters in (6.2) must sum to zero over any of its subscripts.

With these constraints, the effects parameters can be expressed in terms of averages of the model means. For example,

$$\begin{aligned}
\mu &= \bar\mu_{\cdot\cdot\cdot} \quad \alpha_i = \bar\mu_{i\cdot\cdot} - \bar\mu_{\cdot\cdot\cdot} \quad \beta_j = \bar\mu_{\cdot j\cdot} - \bar\mu_{\cdot\cdot\cdot} \quad \gamma_k = \bar\mu_{\cdot\cdot k} - \bar\mu_{\cdot\cdot\cdot}, \\
(\alpha\beta)_{ij} &= \bar\mu_{ij\cdot} - \bar\mu_{i\cdot\cdot} - \bar\mu_{\cdot j\cdot} + \bar\mu_{\cdot\cdot\cdot}, \qquad \text{etc.,} \\
(\alpha\beta\gamma)_{ijk} &= \mu_{ijk} - \bar\mu_{ij\cdot} - \bar\mu_{i\cdot k} - \bar\mu_{\cdot jk} \\
&\quad + \bar\mu_{i\cdot\cdot} + \bar\mu_{\cdot j\cdot} + \bar\mu_{\cdot\cdot k} - \bar\mu_{\cdot\cdot\cdot}.
\end{aligned} \tag{6.3}$$

Note that main-effect parameters are simply the differences between the model means for one level of a factor and the overall model mean. The two-factor interaction parameters are the differences in the main effects for one level of one of the factors at a fixed level of a second and the main effect of the first factor:

$$\begin{aligned}
(\alpha\beta)_{ij} &= (\bar\mu_{ij\cdot} - \bar\mu_{\cdot j\cdot}) - (\bar\mu_{i\cdot\cdot} - \bar\mu_{\cdot\cdot\cdot}) \\
&= (\text{main effect for } A \text{ at level } j \text{ of } B) \\
&\quad - (\text{main effect for } A).
\end{aligned}$$

The three-factor interaction can be similarly viewed as the difference in the two-factor interaction between any two of the factors, say A and B, at a fixed level of the third factor, say C, and the two-factor interaction between A and B:

$$\begin{aligned}
(\alpha\beta\gamma)_{ijk} &= (\mu_{ijk} - \bar\mu_{i\cdot k} - \bar\mu_{\cdot jk} + \bar\mu_{\cdot\cdot k}) \\
&\quad - (\bar\mu_{ij\cdot} - \bar\mu_{i\cdot\cdot} - \bar\mu_{\cdot j\cdot} + \bar\mu_{\cdot\cdot\cdot}) \\
&= (A \times B \text{ interaction at level } k \text{ of } C) \\
&\quad - (A \times B \text{ interaction}).
\end{aligned}$$

It is important to note that while the means in the model (6.1) are uniquely defined, main-effect and interaction parameters can be introduced in many ways; that is they need not be defined as in (6.2). The form (6.2) is used in this book explicitly to facilitate an understanding of inference procedures for multifactor experiments.

Even though the main-effect and interaction parameters can be defined in many ways, certain functions of them are uniquely defined in terms of the model means. The functions that are uniquely defined are precisely those that allow comparison among factor–level effects. We shall return to this topic in Section 6.2.

When considering whether to include interaction terms in a statistical model, one convention that is adopted is to use only hierarchical factor effects. A model is hierarchical (see Exhibit 6.4) if any interaction term involving k factors is included in the specification of the model only if the main effects and all lower-order interactions involving two, three, $\ldots, k-1$ of the factors are also included in the model. In this way, a high-order interaction term is only added to the model if the main effects and lower-order interactions are not able to satisfactorily model the response.

EXHIBIT 6.4

Hierarchical Model. A statistical model is hierarchical if an interaction term involving k factors is included only when the main effects and all lower-order interaction terms involving the k factors are also included in the model.

We return to analysis-of-variance models in the next section, where the estimation of the model parameters is discussed. We now wish to examine the analysis-of-variance partitioning of the total sum of squares for multifactor experiments. The response variability can be partitioned into components for each of the main effects and interactions and for random variability.

6.1.3 Analysis-of-Variance Tables

The pilot-plant experiment shown in Table 6.1 is balanced, because each of the factor–level combinations appears twice in the experiment. We shall use this example and the three-factor model (6.1) to illustrate the construction of analysis-of-variance tables for complete factorial experiments from balanced designs. The three-factor model is sufficiently complex to demonstrate the general construction of main effects and interaction sums of squares.

Generalize the model (6.1) to three factors, of which factor A has a levels (subscript i), factor B has b levels (subscript j), and factor C has c levels (subscript k). Let there be r repeat tests for each combination of the factors. Thus, the experiment consists of a complete factorial experiment in three factors with r repeat tests per factor–level combination.

Throughout earlier chapters of this book the sample variance or its square root, the standard deviation, was used to measure variability. The first step in an analysis-of-variance procedure is to define a suitable measure of variation for which a partitioning into components due to assignable causes and to random variation can be accomplished. While there are many measures that could be used, the numerator of the sample variances is used for a variety of computational and theoretical reasons. This measure of variability is referred to as the *total sum of squares* (TSS):

$$\mathrm{TSS} = \sum_{i=1}^{n} (y_i - \overline{y})^2.$$

The total sum of squares adjusts for the overall average by subtracting it from each individual response; consequently, it is sometimes referred to as the total *adjusted* sum of squares, or TSS(adj). Because we shall always adjust for the sample mean, we drop the word "adjusted" in the name.

For a three-factor experiment, the total sum of squares consists of the squared differences of the observations y_{ijkl} from the overall average response $\overline{y}_{\bullet\bullet\bullet\bullet}$:

$$\mathrm{TSS} = \sum_i \sum_j \sum_k \sum_l (y_{ijkl} - \overline{y}_{\bullet\bullet\bullet\bullet})^2. \tag{6.4}$$

To partition the above total sum of squares into components for the assignable causes (factors A, B, C) and for random (uncontrolled experimental) variation, add and subtract the factor–level combination averages $\overline{y}_{ijk\bullet}$ from each term in the summation for the total sum of squares. Then

$$\begin{aligned}
\mathrm{TSS} &= \sum_i \sum_j \sum_k \sum_l (y_{ijkl} - \overline{y}_{ijk\bullet} + \overline{y}_{ijk\bullet} - \overline{y}_{\bullet\bullet\bullet})^2 \\
&= r \sum_i \sum_j \sum_k (\overline{y}_{ijk\bullet} - \overline{y}_{\bullet\bullet\bullet})^2 + \sum_i \sum_j \sum_k \sum_l (y_{ijkl} - \overline{y}_{ijk\bullet})^2 \qquad (6.5) \\
&= \mathrm{MSS} + \mathrm{SS}_E.
\end{aligned}$$

The first component of (6.5) is the contribution to the total sum of squares of all the variability attributable to assignable causes, often called the *model* sum of squares:

$$\mathrm{MSS} = r \sum_i \sum_j \sum_k (\overline{y}_{ijk\bullet} - \overline{y}_{\bullet\bullet\bullet})^2.$$

The model sum of squares can be partitioned into the following components:

$$\mathrm{MSS} = \mathrm{SS}_A + \mathrm{SS}_B + \mathrm{SS}_C + \mathrm{SS}_{AB} + \mathrm{SS}_{AC} + \mathrm{SS}_{BC} + \mathrm{SS}_{ABC}, \tag{6.6}$$

where SS_A, SS_B, and SS_C measure the individual factor contributions (main effects of the three experimental factors) to the total sum of squares, SS_{AB}, SS_{AC}, and SS_{BC} measure the contributions of the two-factor joint effects, and SS_{ABC} measures the contribution of the joint effects of all three experimental factors above the contributions measured by the main effects and the two-factor interactions.

The sums of squares for the factor main effects are multiples of the sums of the squared differences between the averages for each level of the factor and the overall average; for example,

$$\mathrm{SS}_A = bcr \sum_{i=1}^{a} (\bar{y}_{i\cdot\cdot\cdot} - \bar{y}_{\cdot\cdot\cdot\cdot})^2. \tag{6.7}$$

The differences between the averages for each level of a factor and the overall average yields a direct measure of the individual factor– level effects, the main effects of the factor. The sum of squares (6.7) is a composite measure of the combined effects of the factor levels on the response. Note that the multiplier in front of the summation sign is the total number of observations that are used in the calculation of the averages $\bar{y}_{i\cdot\cdot\cdot}$ for each level of factor A. Sums of squares for each of the main effects are calculated as in (6.7). (Note: computational formulas are shown below in Table 6.2.)

Just as the main effect for a factor can be calculated by taking differences between individual averages and the overall average, interaction effects can be calculated by taking differences between (a) the main effects of one factor at a particular level of a second factor and (b) the overall main effect of the factor. For example, the interaction effects for factors A and B can be obtained by calculating:

$$\begin{aligned}
&(\text{Main effect for } A \text{ at level } j \text{ of } B) - (\text{main effect for } A) \\
&\quad = (\bar{y}_{ij\cdot\cdot} - \bar{y}_{\cdot j\cdot\cdot}) - (\bar{y}_{i\cdot\cdot\cdot} - \bar{y}_{\cdot\cdot\cdot\cdot}) \\
&\quad = \bar{y}_{ij\cdot\cdot} - \bar{y}_{i\cdot\cdot\cdot} - \bar{y}_{\cdot j\cdot\cdot} + \bar{y}_{\cdot\cdot\cdot\cdot}.
\end{aligned}$$

This is the same expression one would obtain by reversing the roles of the two factors. The interaction effects can be combined in an overall sum of squares for the interaction effects of factors A and B:

$$\mathrm{SS}_{AB} = cr \sum_{i=1}^{a} \sum_{j=1}^{b} (\bar{y}_{ij\cdot\cdot} - \bar{y}_{i\cdot\cdot\cdot} - \bar{y}_{\cdot j\cdot\cdot} + \bar{y}_{\cdot\cdot\cdot\cdot})^2. \tag{6.8}$$

TABLE 6.2 Symbolic Analysis-of-Variance Table for Three Factors

ANOVA

Source of Variation	Degrees of Freedom (df)	Sum of Squares	Mean Square	F-Value
A	$a-1$	SS_A	$MS_A = SS_A/\text{df}(A)$	$F_A = MS_A/MS_E$
B	$b-1$	SS_B	$MS_B = SS_B/\text{df}(B)$	$F_B = MS_B/MS_E$
C	$c-1$	SS_C	$MS_C = SS_C/\text{df}(C)$	$F_C = MS_C/MS_E$
AB	$(a-1)(b-1)$	SS_{AB}	$MS_{AB} = SS_{AB}/\text{df}(AB)$	$F_{AB} = MS_{AB}/MS_E$
AC	$(a-1)(c-1)$	SS_{AC}	$MS_{AC} = SS_{AC}/\text{df}(AC)$	$F_{AC} = MS_{AC}/MS_E$
BC	$(b-1)(c-1)$	SS_{BC}	$MS_{BC} = SS_{BC}/\text{df}(BC)$	$F_{BC} = MS_{BC}/MS_E$
ABC	$(a-1)(b-1)(c-1)$	SS_{ABC}	$MS_{ABC} = SS_{ABC}/\text{df}(ABC)$	$F_{ABC} = MS_{ABC}/MS_E$
Error	$abc(r-1)$	SS_E	$MS_E = SS_E/\text{df(Error)}$	
Total	$abcr-1$	TSS		

Calculations

$TSS = \sum_i \sum_j \sum_k \sum_l y_{ijkl}^2 - SS_M$ $\quad SS_M = y_{\cdot\cdot\cdot\cdot}^2/n$

$SS_A = \sum_i y_{i\cdot\cdot\cdot}^2/bcr - SS_M$ $\quad SS_{AB} = \sum_i \sum_j y_{ij\cdot\cdot}^2/cr - SS_M - SS_A - SS_B$

$SS_B = \sum_j y_{\cdot j\cdot\cdot}^2/acr - SS_M$ $\quad SS_{AC} = \sum_i \sum_k y_{i\cdot k\cdot}^2/br - SS_M - SS_A - SS_C$

$SS_C = \sum_k y_{\cdot\cdot k\cdot}^2/abr - SS_M$ $\quad SS_{BC} = \sum_j \sum_k y_{\cdot jk\cdot}^2/ar - SS_M - SS_B - SS_C$

$SS_{ABC} = \sum_i \sum_j \sum_k y_{ijk\cdot}^2/r - SS_M - SS_A - SS_B - SS_C - SS_{AB} - SS_{AC} - SS_{BC}$

$SS_E = TSS - SS_A - SS_B - SS_C - SS_{AB} - SS_{AC} - SS_{BC} - SS_{ABC}$

$y_{\cdot\cdot\cdot\cdot} = \sum_i \sum_j \sum_k \sum_l y_{ijkl}$ $\quad y_{i\cdot\cdot\cdot} = \sum_j \sum_k \sum_l y_{ijkl}$ $\quad n = abcr$

$y_{ij\cdot\cdot} = \sum_k \sum_l y_{ijkl}$ $\quad y_{ijk\cdot} = \sum_l y_{ijkl}$ $\quad \text{df}(A) = a-1$, etc.

The sum of squares for the three-factor interaction, SS_{ABC}, can be obtained in a similar fashion by examining the differences between the interaction effects for two of the factors at fixed levels of the third factor and the overall interaction effect of the two factors. The result of combining these three-factor interaction effects is

$$SS_{ABC} = r \sum_{i=1}^{a} \sum_{j=1}^{b} \sum_{k=1}^{c} (\overline{y}_{ijk\bullet} - \overline{y}_{ij\bullet\bullet} - \overline{y}_{i\bullet k\bullet} - \overline{y}_{\bullet jk\bullet} + \overline{y}_{i\bullet\bullet\bullet} + \overline{y}_{\bullet j\bullet\bullet} + \overline{y}_{\bullet\bullet k\bullet} - \overline{y}_{\bullet\bullet\bullet\bullet})^2. \tag{6.9}$$

The final term needed to complete the partitioning of the total sum of squares is the second component of (6.5), the error sum of squares, SS_E. Algebraically, the error sum of squares is simply the sum of the squared differences between the individual repeat-test responses and the averages for the factor–level combinations:

$$SS_E = \sum_{i=1}^{a} \sum_{j=1}^{b} \sum_{k=1}^{c} \sum_{l=1}^{r} (y_{ijkl} - \overline{y}_{ijk\bullet})^2. \tag{6.10}$$

The information about factor effects that is obtained from the partitioning of the total sum of squares is summarized in an analysis-of-variance table. A symbolic ANOVA table for the three-factor interaction model we have been considering is shown in Table 6.2. In this table, the first column, "source of variation," designates the partitioning of the response variability into the various components included in the statistical model.

The second column, "degrees of freedom," partitions the sample size into similar components that relate the amount of information obtained on each factor, the interaction, and the error term of the model. Note that each main effect has one less degree of freedom than the number of levels of the factor. Interaction degrees of freedom are conveniently calculated as products of the degrees of freedom of the respective main effects. The number of error degrees of freedom is the number of factor – level combinations multiplied by one less than the number of repeat tests for each combination. These degrees of freedom can all be found by identifying constraints on the terms in each of the sums of squares.

The degrees of freedom indicate how many statistically independent response variables or functions of the response variables comprise a sum of squares. For example, the $n = abcr$ response variables y_{ijlk} in a three-factor balanced complete factorial experiment are assumed to be statistically independent (Exhibit 6.3). Consequently the *unadjusted* total sum of squares

$$\sum_i \sum_j \sum_k \sum_l y_{ijkl}^2$$

has n degrees of freedom. The total *adjusted* sum of squares (6.3) does not have n degrees of freedom because there is a constraint on the n terms in the sum of squares;

$$\sum_i \sum_j \sum_k \sum_l (y_{ijkl} - \bar{y}_{\bullet\bullet\bullet\bullet}) = 0.$$

Hence, there are $abcr - 1$ degrees of freedom associated with the sum of squares TSS.

Similarly, the error sum of squares SS_E does not have n degrees of freedom, even though there are n terms in its sum of squares. For each combination of the factor levels,

$$\sum_l (y_{ijkl} - \bar{y}_{ijk\bullet}) = 0.$$

Thus the r deviations $y_{ijkl} - \bar{y}_{ijk\bullet}$ have $r - 1$ degrees of freedom. Because this is true for each of the factor–level combinations, the error sum of squares has $abc(r - 1)$ degrees of freedom. In a similar fashion, each of the main effect and interaction degrees of freedom can be calculated by determining the constraints on the terms in each sum of squares. The results are the degrees of freedom listed in Table 6.2.

The third column of Table 6.2 contains the sums of squares for various main effects and interactions, and the error components of the model. The fourth column, "mean squares," contains the respective sums of squares divided by their numbers of degrees of freedom. These statistics are used for forming the F-ratios in the next column, each main effect and interaction mean square being divided by the error mean square.

Table 6.3 lists summary statistics used in the calculation of the sums of squares for the pilot-plant chemical-yield experiment. The sums of squares can be calculated using these statistics and the computational formulas shown in Table 6.2. The complete ANOVA table is displayed in Table 6.4. Several of the F-statistics in the table are much larger than 1. The interpretation of these large F-values will be discussed in Section 6.3.

Figure 6.1 is a schematic diagram of an experiment that was conducted to study flow rates (cc/min) of a particle slurry that was pumped under constant pressure for a fixed time period through a cylindrical housing. The factor of interest in this experiment is the type of filter placed in the housing. The test sequence was randomized so that the experiment was conducted using a completely randomized design. The purpose of the experiment was to assess the effects, if any, of the filter types on the average flow rates.

TABLE 6.3 Summary Statistics for Pilot-Plant Chemical-Yield Experiment

Factor		
Symbol	Subscript	Label
A	i	Temperature
B	j	Concentration
C	k	Catalyst

$$\sum_i\sum_j\sum_k\sum_l y_{ijkl}^2 = 68,748 \quad y_{\bullet\bullet\bullet\bullet}^2 = (1028)^2 = 1,056,784$$

$$\sum_i y_{i\bullet\bullet\bullet}^2 = (422)^2 + (606)^2 = 545,320$$

$$\sum_j y_{\bullet j\bullet\bullet}^2 = (534)^2 + (494)^2 = 529,192$$

$$\sum_k y_{\bullet\bullet k\bullet}^2 = (508)^2 + (520)^2 = 528,464$$

$$\sum_i\sum_j y_{ij\bullet\bullet}^2 = (224)^2 + (198)^2 + (310)^2 + (296)^2 = 273,096$$

$$\sum_i\sum_k y_{i\bullet k\bullet}^2 = (228)^2 + (194)^2 + (280)^2 + (326)^2 = 274,296$$

$$\sum_j\sum_k y_{\bullet jk\bullet}^2 = (264)^2 + (270)^2 + (244)^2 + (250)^2 = 264,632$$

$$\begin{aligned}\sum_i\sum_j\sum_k y_{ijk\bullet}^2 &= (120)^2 + (104)^2 + (108)^2 + (90)^2 \\ &\quad + (144)^2 + (166)^2 + (136)^2 + (160)^2 = 137,368\end{aligned}$$

TABLE 6.4 ANOVA Table for Pilot-Plant Chemical-Yield Study

Source of Variation	df	Sum of Squares	Mean Square	F-Value
Temperature T	1	2116.00	2116.00	264.50
Concentration Co	1	100.00	100.00	12.50
Catalyst Ca	1	9.00	9.00	1.13
$T \times$ Co	1	9.00	9.00	1.13
$T \times$ Ca	1	400.00	400.00	50.00
Co $\times$ Ca	1	0.00	0.00	0.00
$T \times$ Co $\times$ Ca	1	1.00	1.00	0.13
Error	8	64.00	8.00	
Total	15	2699.00		

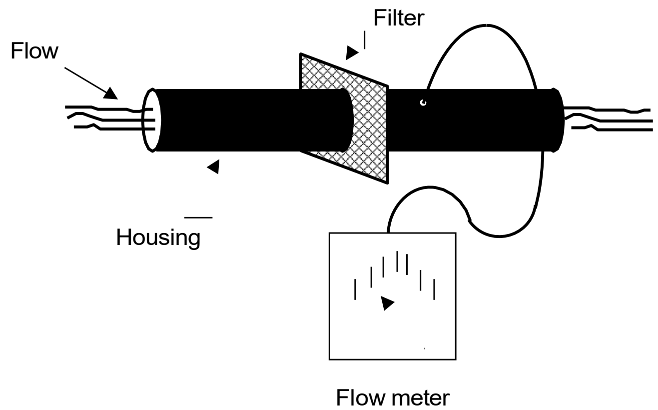

Figure 6.1 Flow-rate experiment.

The only factor in this experiment is the filter type chosen for each test run. This factor has four levels. Because inferences are desired on these specific filter types, the factor has fixed effects. If only two filter types were included in the experiment, the two-sample inference procedures detailed in Chapter 3 could be used to assess the filter effects on the flow rates. To assess the influence of all four filter types, the inferential techniques just discussed must be used to accommodate more than two factor levels (equivalently, more than two population or process means).

TABLE 6.5 Analysis of Variance Table for Flow-Rate Data

ANOVA

Source of Variation	Degrees of Freedom (df)	Sum of Squares	Mean Square	F-Value
Filter	3	0.00820	0.00273	1.86
Error	12	0.01765	0.00147	
Total	15	0.02585		

Data

Filter	Flow Rates				Total
A	0.233	0.197	0.259	0.244	0.933
B	0.259	0.258	0.343	0.305	1.165
C	0.183	0.284	0.264	0.258	0.989
D	0.233	0.328	0.267	0.269	1.097
Total					4.184

The partitioning of the total sum of squares is summarized in the ANOVA table, Table 6.5. This special case of ANOVA for single-factor experiments is often termed a *one-way* analysis of variance and the corresponding ANOVA model a *one-way classification model*. In this case the F-statistic generalizes the two sample t -statistic (Equation 3.14) to comparisons of more than two population or process means. The F-statistic for this example is not apprecia- bly larger than 1. In Section 6.3 we shall see how to interpret this finding.

6.2 PARAMETER ESTIMATION

Inferences on specific factor effects requires the estimation of the parameters of ANOVA models. In Section 6.2.1 estimation of the error standard deviation is discussed. In Section 6.2.2 the estimation of both the means of fixed-effects models and parameters representing main effects and interactions are dis- cussed. Relationships are established between estimates of the means of the fixed-effects models and estimates of the parameters representing the main effects and interactions. Alternative estimates of factor effects for quantitative factor levels are presented in Section 6.2.3.

6.2.1 Estimation of the Error Standard Deviation

A key assumption in the modeling of responses from designed experiments is that the experimental conditions are sufficiently stable that the model errors can be considered to follow a common probability distribution. Ordinarily the distribution assumed for the errors is the normal probability distribution. There are several justifications for this assumption.

First, the model diagnostics discussed in Chapter 18 can be used to assess whether this assumption is reasonable. Quantile plots, similar to those dis- cussed in Sections 5.4 and 18.2, provide an important visual assessment of the reasonableness of the assumption of normally distributed errors. Second, fixed-effects models are extensions of the one- or two-sample experiments discussed in Chapter 3. The central limit property (Section 2.3) thus provides a justification for the use of normal distributions for averages and functions of averages that constitute factor effects even if the errors are not normally distributed. Finally, randomization provides a third justification: the process of randomization induces a sampling distribution on factor effects that is closely approximated by the theoretical distributions of the t and F statistics used to make inferences on the parameters of ANOVA models.

Assuming that the errors follow a common distribution, the error mean square from the ANOVA table is a pooled estimator of the common error variance. If the model consists of all main effect and interaction parameters (a *saturated* model) and the experiment is a complete factorial with repeat tests, the error sum of squares is similar in form to equation (6.10) for three factors. With the balance in the design, the error mean square is simply the average of the sample variances calculated from the repeat tests for each factor– level combination:

$$s_e^2 = \mathrm{MS}_E = \frac{1}{abc}\sum_i \sum_j \sum_k s_{ijk}^2. \tag{6.11}$$

If there are no repeat tests and the model is saturated, there is no estimate of experimental error available. Equation (6.11) is not defined because none of the sample variances can be calculated. In this situation an estimate of experimental-error variation is only obtainable if some of the parameters in the saturated model can be assumed to be zero.

To illustrate why estimates of experimental-error variation are available when certain model parameters are assumed to be zero, consider model (6.1). When there is a single factor, $y_{ij} = \mu + a_i + e_{ij}$. If the factor levels all have the same constant effect on the response, all the a_i in the model are zero. Then $\bar{y}_{i\cdot} = \mu + \bar{e}_{i\cdot}$ and $\bar{y}_{\cdot\cdot} = \mu + \bar{e}_{\cdot\cdot}$, so that the main effect for factor level i is

$$\bar{y}_{i\cdot} - \bar{y}_{\cdot\cdot} = \bar{e}_{i\cdot} - \bar{e}_{\cdot\cdot}.$$

Thus, SS_A [equation (6.7)] does not estimate any function of the main effect parameters. It measures variability in the averages of the errors. One can show that both MS_A and MS_E in this situation are estimators of the error variance σ^2.